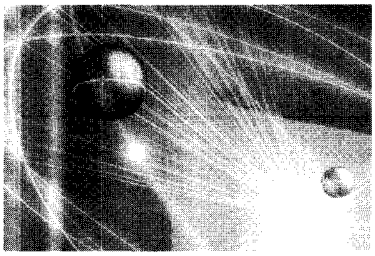


CHAPTER 8

Conics, Parametric Curves, and Polar Curves

Introduction Until now, most curves we have encountered have been graphs of functions, and they provided useful visual information about the behaviour of the functions. In this chapter we begin to look at plane curves as interesting objects in their own right. First, we examine conic sections, curves with quadratic equations obtained by intersecting a plane with a right-circular cone. Then we consider curves that can be described by two parametric equations that give the coordinates of points on the curve as functions of a parameter. If this parameter is time, the equations describe the path of a moving point in the plane. Finally, we consider curves described by equations in a new coordinate system called polar coordinates, in which a point is located by giving its distance and direction from the origin. In Chapter 11 we will expand our study of curves to three dimensions.

8.1 Conics

Circles, ellipses, parabolas, and hyperbolas are called **conic sections** (or, more simply, just **conics**) because they are curves in which planes intersect right-circular cones.

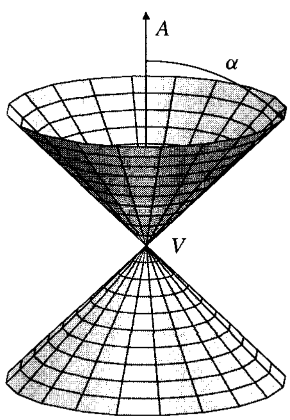


Figure 8.1 A cone with vertex V, axis A, and semi-vertical angle α

To be specific, suppose that a line A is fixed in space, and V is a point fixed on A. The **right-circular cone** having **axis** A, **vertex** V, and **semi-vertical angle** α is the surface consisting of all points on straight lines through V that make angle α with the line A. (See Figure 8.1.) The cone has two halves (called **nappes**) lying on opposite sides of the vertex V. Any plane P that does not pass through V will intersect the cone (one or both nappes) in a curve C. (See Figure 8.2.) If a line normal (i.e., perpendicular) to P makes angle θ with the axis A of the cone, where $0 \le \theta \le \pi/2$, then

$$\begin{array}{ll} C \text{ is a circle if} & \theta = 0 \\ C \text{ is an ellipse if} & 0 < \theta < \dfrac{\pi}{2} - \alpha \\ C \text{ is a parabola if} & \theta = \dfrac{\pi}{2} - \alpha \\ C \text{ is a hyperbola if} & \dfrac{\pi}{2} - \alpha < \theta \le \dfrac{\pi}{2}. \end{array}$$

In Sections 10.4 and 10.5 it is shown that planes are represented by first-degree equations and cones by second-degree equations. Therefore, all conics can be represented analytically (in terms of Cartesian coordinates x and y in the plane of the conic) by a second-degree equation of the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

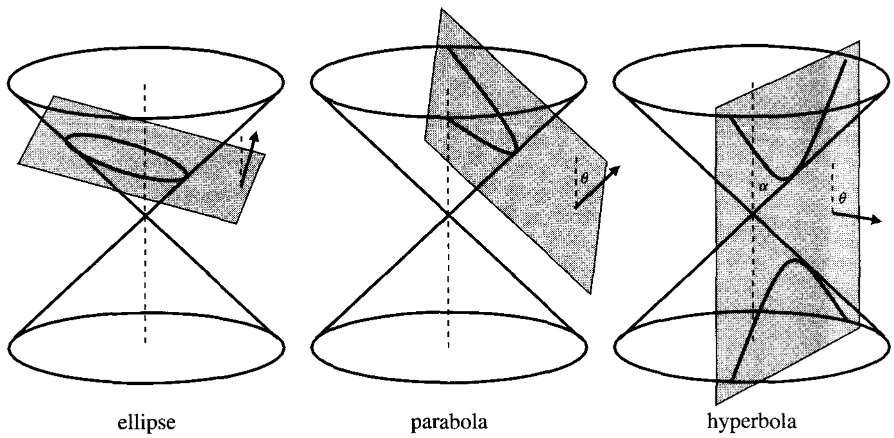


Figure 8.2 Planes intersecting cones in an ellipse, a parabola, and a hyperbola

where $A, B, \ldots, F$ are constants. However, such an equation can also represent the empty set, a single point, or one or two straight lines if the left-hand side factors into linear factors:

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0.$$

After straight lines the conic sections are the simplest of plane curves. They have many properties that make them useful in applications of mathematics; that is why we include a discussion of them here. Much of this material is optional from the point of view of a calculus course, but familiarity with the properties of conics can be very important in some applications. Most of the properties of conics were discovered by the Greek geometer Apollonius of Perga, around 200 BC. It is remarkable that he was able to obtain these properties using only the techniques of classical Euclidean geometry; today, most of these properties are expressed more conveniently using analytic geometry and specific coordinate systems.

Parabolas

DEFINITION 1

Parabolas

A **parabola** consists of points in the plane that are equidistant from a given point (the **focus** of the parabola) and a given straight line (the **directrix** of the parabola). The line through the focus perpendicular to the directrix is called the **principal axis** (or simply **the axis**) of the parabola. The **vertex** of the parabola is the point where the parabola crosses its principal axis. It is on the axis halfway between the focus and the directrix.

Example 1 Find an equation of the parabola whose focus is the point $F = (a, 0)$ and whose directrix is the line L with equation $x = -a$.

Solution The parabola has axis along the x-axis and vertex at the origin. (See Figure 8.3.) If $P = (x, y)$ is any point on the parabola, then the distance from P to

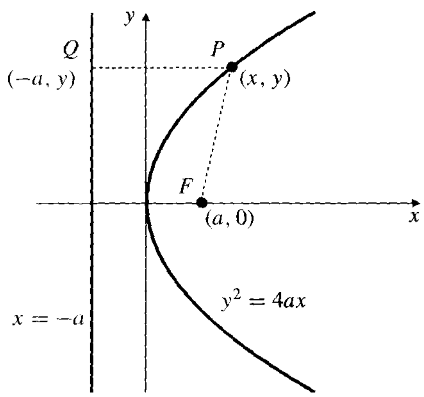


Figure 8.3 $PF = PQ$: the defining property of a parabola

F is equal to the distance from P to the nearest point Q on L. Thus

$$\sqrt{(x-a)^2 + y^2} = x + a$$
$$\text{or } x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2,$$

or, upon simplification, $y^2 = 4ax$. ∎

Similarly, we can obtain standard equations for parabolas with vertices at the origin and foci at $(-a, 0)$, $(0, a)$ and $(0, -a)$:

Table 1. Standard equations of parabolas

Focus	Directrix	Equation
$(a, 0)$	$x = -a$	$y^2 = 4ax$
$(-a, 0)$	$x = a$	$y^2 = -4ax$
$(0, a)$	$y = -a$	$x^2 = 4ay$
$(0, -a)$	$y = a$	$x^2 = -4ay$

The Focal Property of a Parabola

All of the conic sections have interesting and useful focal properties relating to the way in which surfaces of revolution they generate reflect light if the surfaces are mirrors. For instance, a circle will clearly reflect back along the same path any ray of light incident along a line passing through its centre. The focal properties of parabolas, ellipses, and hyperbolas can be derived from the reflecting property of a straight line (i.e., a plane mirror) by elementary geometrical arguments.

Light travels in straight lines in a medium of constant optical density (one where the speed of light is constant). This is a consequence of the physical Principle of Least Action, which asserts that in travelling between two points, light takes the path requiring the minimum travel time. Given a straight line L in a plane and two points A and B in the plane on the same side of L, the point P on L for which the sum of the distances $AP + PB$ is minimum is such that AP and PB make equal angles with L, or equivalently, with the normal to L at P. (See Figure 8.4.) If B' is the point such that L is the right bisector of the line segment BB', then P is the intersection of L and AB'. Since one side of a triangle cannot exceed the sum of the other two sides,

$$AP + PB = AP + PB' = AB' \le AQ + QB' = AQ + QB.$$

Reflection by a straight line

The point P on L at which a ray from A reflects so as to pass through B is the point that minimizes the sum of the distances $AP + PB$.

Now consider a parabola with focus F and directrix D. Let P be on the parabola and let T be the line tangent to the parabola at P. (See Figure 8.5.) Let Q be any point on T. Then FQ meets the parabola at a point X between F and Q. Let M and N be points on D such that MX and NP are perpendicular to D, and let A be a point on the line through N and P that lies on the same side of the parabola as F. We have

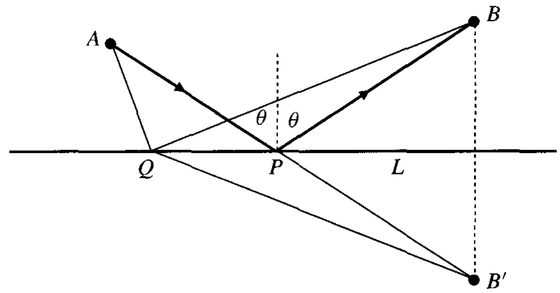


Figure 8.4 Reflection by a straight line

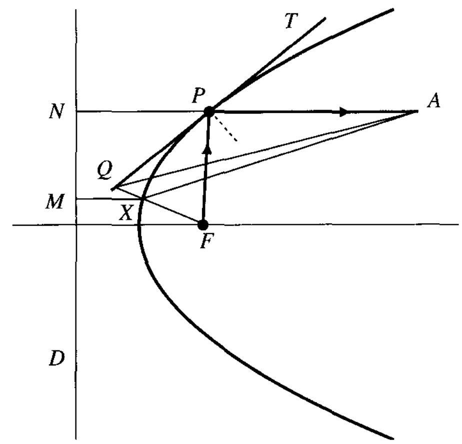


Figure 8.5 Reflection by a parabola

BEWARE
Consider the equalities and inequalities in this chain one at a time. Why is each one true?

$$\begin{aligned} FP + PA = NP + PA = NA &\le MX + XA = FX + XA \\ &\le FX + XQ + QA = FQ + QA. \end{aligned}$$

Thus, among all points Q on the line T, $Q = P$ is the one that minimizes the sum of distances $FQ + QA$. By the observation made for straight lines above, FP and PA make equal angles with T and so also with the normal to the parabola at P. (The parabola and the tangent line have the same normal at P.)

Reflection by a parabola

Any ray from the focus will be reflected parallel to the axis of the parabola. Equivalently, any incident ray parallel to the axis of the parabola will be reflected through the focus.

Ellipses

DEFINITION 2

Ellipses

An **ellipse** consists of all points in the plane, the sum of whose distances from two fixed points (the **foci**) is constant.

Example 2 Find the ellipse with foci at the points $(-c, 0)$ and $(c, 0)$ if the sum of the distances from any point P on the ellipse to these two foci is $2a$.

Solution The ellipse passes through the four points $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$, where $b^2 = a^2 - c^2$. (See Figure 8.6.) Also, if $P = (x, y)$ is on the ellipse, then

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing one term from the left side to the right side and squaring, we get

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

Now we expand the squares, cancel terms, transpose, and square again:

$$\begin{aligned} a\sqrt{(x+c)^2+y^2} &= a^2+cx \\ a^2(x^2+2cx+c^2+y^2) &= a^4+2a^2cx+c^2x^2 \\ (a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2). \end{aligned}$$

Finally, replace a^2-c^2 with b^2 and divide by a^2b^2 to get the standard equation of the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

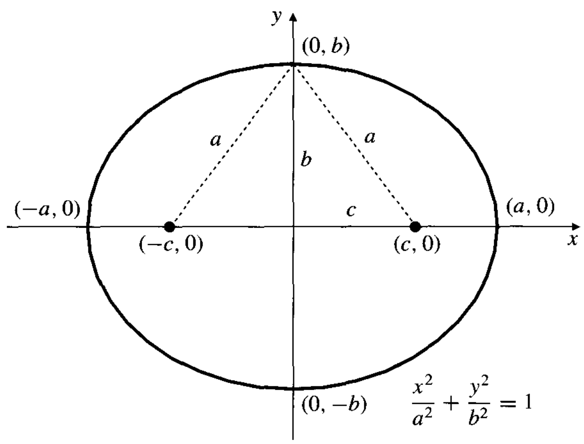


Figure 8.6 An ellipse

The following quantities describe this ellipse:

a is the **semi-major axis**
b is the **semi-minor axis**
$c=\sqrt{a^2-b^2}$ is the **semi-focal separation**.

The point halfway between the foci is called the **centre** of the ellipse. In the example above it is the origin. Note that $a>b$ in this example. If $a<b$, then the ellipse has its foci at $(0, c)$ and $(0, -c)$, where $c=\sqrt{b^2-a^2}$. The line containing the foci (the **major axis**) and the line through the centre perpendicular to that line (the **minor axis**) are called the **principal axes** of the ellipse.

The **eccentricity** of an ellipse is the ratio of the semi-focal separation to the semi-major axis. We denote the eccentricity ε. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$,

$$\varepsilon=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}.$$

Note that $\varepsilon<1$ for any ellipse; the greater the value of ε, the more elongated (less circular) is the ellipse. If $\varepsilon=0$ so that $a=b$ and $c=0$, the two foci coincide and the ellipse is a circle.

The Focal Property of an Ellipse

Let P be any point on an ellipse having foci F_1 and F_2. The normal to the ellipse at P bisects the angle between the lines F_1P and F_2P.

> **Reflection by an ellipse**
>
> Any ray coming from one focus of an ellipse will be reflected through the other focus.

To see this, observe that if Q is any point on the line T tangent to the ellipse at P, then F_1Q meets the ellipse at a point X between F_1 and Q (see Figure 8.7), so

$$F_1P + PF_2 = F_1X + XF_2 \le F_1X + XQ + QF_2 = F_1Q + QF_2.$$

Among all points on T, P is the one that minimizes the sum of the distances to F_1 and F_2. This implies that the normal to the ellipse at P bisects the angle F_1PF_2.

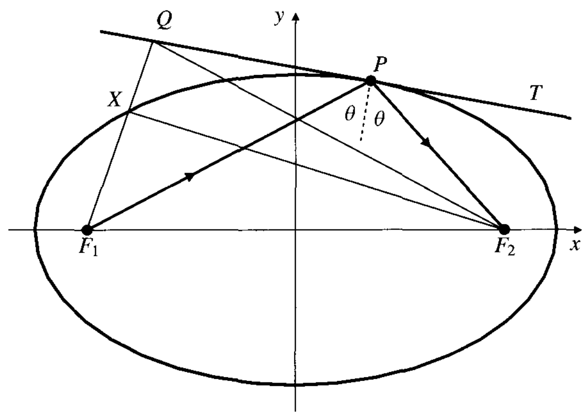


Figure 8.7 A ray from one focus of an ellipse is reflected to the other focus

The Directrices of an Ellipse

If $a > b > 0$, each of the lines $x = a/\varepsilon$ and $x = -a/\varepsilon$ is called a **directrix** of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. If P is on the ellipse, then the ratio of the distance from P to a focus to its distance from the corresponding directrix is equal to the eccentricity ε. If $P = (x, y)$, F is the focus $(c, 0)$, Q is on the corresponding directrix $x = a/\varepsilon$, and PQ is perpendicular to the directrix, then (see Figure 8.8)

$$\begin{aligned}
PF^2 &= (x-c)^2 + y^2 \\
&= x^2 - 2cx + c^2 + b^2\left(1 - \frac{x^2}{a^2}\right) \\
&= x^2\left(\frac{a^2-b^2}{a^2}\right) - 2cx + a^2 - b^2 + b^2 \\
&= \varepsilon^2x^2 - 2\varepsilon ax + a^2 \qquad \text{(because } c = \varepsilon a). \\
&= (a - \varepsilon x)^2.
\end{aligned}$$

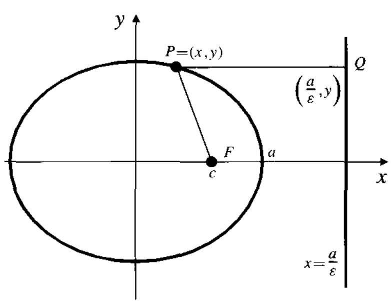


Figure 8.8 A focus and corresponding directrix of an ellipse

Thus $PF = a - \varepsilon x$. Also, $QP = (a/\varepsilon) - x = (a - \varepsilon x)/\varepsilon$. Therefore $PF/QP = \varepsilon$, as asserted.

A parabola may be considered as the limiting case of an ellipse whose eccentricity has increased to 1. The distance between the foci is infinite, so the centre, one focus, and its corresponding directrix have moved off to infinity leaving only one focus and its directrix in the finite plane.

Hyperbolas

DEFINITION 3

Hyperbolas

A **hyperbola** consists of all points in the plane, the difference of whose distances from two fixed points (the **foci**) is constant.

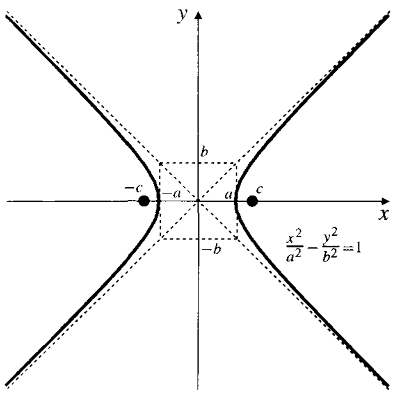


Figure 8.9

Example 3 If the foci of a hyperbola are $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, and the difference of the distances from a point $P = (x, y)$ on the hyperbola to these foci is $2a$ (where $a < c$), then

$$PF_2 - PF_1 = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \begin{cases} 2a & \text{(right branch)} \\ -2a & \text{(left branch).} \end{cases}$$

(See Figure 8.9.) Simplifying this equation by squaring and transposing as was done for the ellipse in Example 2, we obtain the standard equation for the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = c^2 - a^2$. ■

The points $(a, 0)$ and $(-a, 0)$ (called the **vertices**) lie on the hyperbola, one on each branch. (The two branches correspond to the intersections of the plane of the hyperbola with the two nappes of a cone.) Some parameters used to describe the hyperbola are

a	the **semi-transverse axis**
b	the **semi-conjugate axis**
$c = \sqrt{a^2 + b^2}$	the **semi-focal separation.**

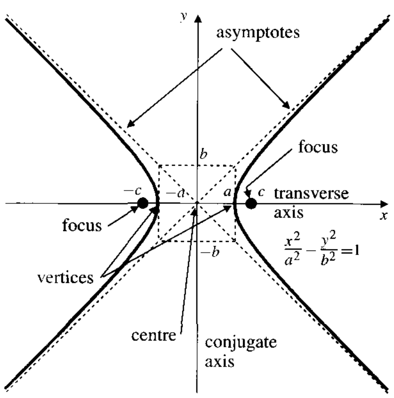


Figure 8.10 Terms associated with a hyperbola

The midpoint of the line segment F_1F_2 (in this case the origin) is called the **centre** of the hyperbola. The line through the centre, the vertices, and the foci is the **transverse axis**. The line through the centre perpendicular to the transverse axis is the **conjugate axis**. The conjugate axis does not intersect the hyperbola. If a rectangle with sides $2a$ and $2b$ is drawn centred at the centre of the hyperbola and with two sides tangent to the hyperbola at the vertices, the two diagonal lines of the rectangle are **asymptotes** of the hyperbola. They have equations $(x/a) \pm (y/b) = 0$; that is, they are solutions of the degenerate equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

The hyperbola approaches arbitrarily close to these lines as it recedes from the origin. (See Figure 8.10.) A **rectangular** hyperbola is one whose asymptotes are perpendicular lines. (This is so if $b = a$.)

The eccentricity of the hyperbola is

$$\varepsilon = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

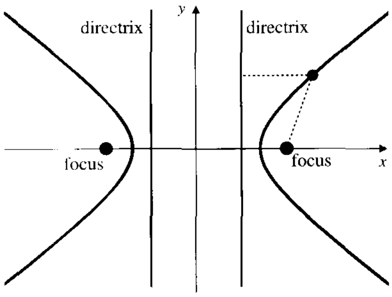


Figure 8.11 The directrices of a hyperbola

Note that $\varepsilon > 1$. The lines $x = \pm(a/\varepsilon)$ are called the **directrices** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. (See Figure 8.11.) In a manner similar to that used for the ellipse, you can show that if P is on the hyperbola, then

$$\frac{\text{distance from } P \text{ to a focus}}{\text{distance from } P \text{ to the corresponding directrix}} = \varepsilon.$$

The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

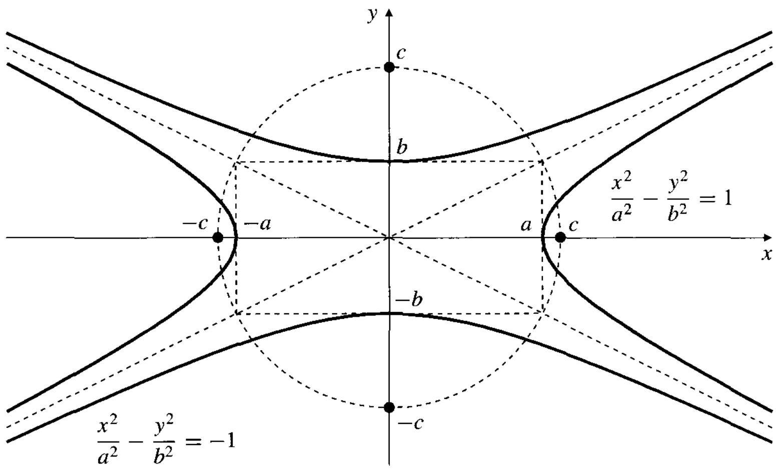


Figure 8.12 Two conjugate hyperbolas and their common asymptotes

A hyperbola with the same asymptotes as $x^2/a^2 - y^2/b^2 = 1$, but with transverse axis along the y-axis, vertices at $(0, b)$ and $(0, -b)$ and foci at $(0, c)$ and $(0, -c)$ is represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

The two hyperbolas are said to be **conjugate** to one another. (See Figure 8.12.) The *conjugate axis* of a hyperbola is the *transverse axis* of the conjugate hyperbola. Together, the transverse and conjugate axes of a hyperbola are called its **principal axes**.

The Focal Property of a Hyperbola

Let P be any point on a hyperbola with foci F_1 and F_2. Then the tangent line to the hyperbola at P bisects the angle between the lines F_1P and F_2P.

Reflection by a hyperbola

A ray from one focus of a hyperbola is reflected by the hyperbola so that it appears to have come from the other focus.

To see this, let P be on the right branch, let T be the line tangent to the hyperbola at P, and let C be a circle of large radius centred at F_2. (See Figure 8.13.) Let F_2P intersect this circle at D. Let Q be any point on T. Then QF_1 meets the hyperbola at X between Q and F_1, and F_2X meets C at E. Since X is on the radial line F_2E,

BEWARE

Check the equalities and inequalities in this chain one at a time to make sure you understand why it is true.

it is closer to E than it is to other points on C. That is, $XE \le XD$. Thus

$$\begin{aligned} F_1P + PD &= F_1P + F_2D - F_2P \\ &= F_2D - (F_2P - F_1P) \\ &= F_2E - (F_2X - F_1X) \\ &= F_1X + F_2E - F_2X \\ &= F_1X + XE \\ &\le F_1X + XD \\ &\le F_1X + XQ + QD = F_1Q + QD. \end{aligned}$$

P is the point on T which minimizes the sum of distances to F_1 and D; therefore the normal to the hyperbola at P bisects the angle F_1PD. Therefore, T bisects the angle F_1PF_2.

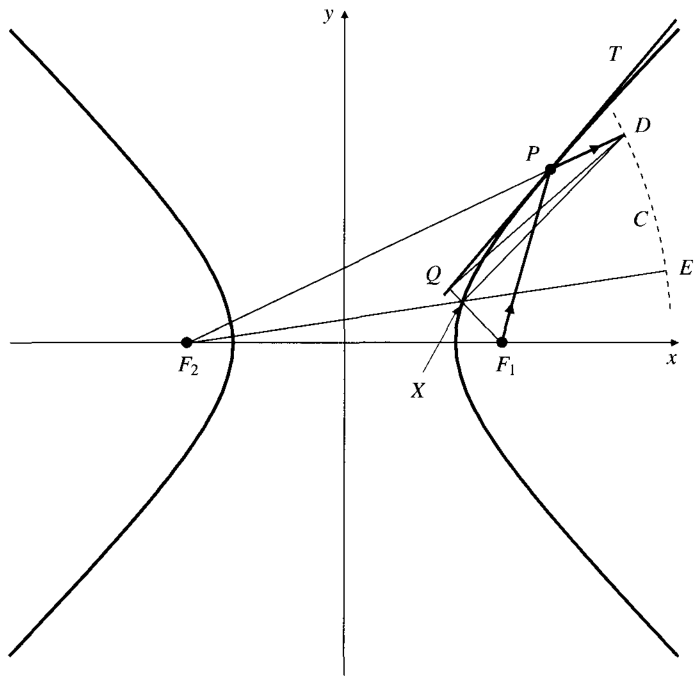


Figure 8.13 A ray from one focus is reflected along a line from the other focus

Classifying General Conics

A second-degree equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (A^2 + B^2 + C^2 > 0),$$

generally represents a conic curve, but in certain degenerate cases it may represent two straight lines ($x^2 - y^2 = 0$ represents the lines $x = y$ and $x = -y$), one straight line ($x^2 = 0$ represents the line $x = 0$), a single point ($x^2 + y^2 = 0$ represents the origin), or no points at all ($x^2 + y^2 = -1$ is not satisfied by any point in the plane).

The nature of the set of points represented by a given second-degree equation can be determined by rewriting the equation in a form that can be recognized as one of the standard types. If $B = 0$, this rewriting can be accomplished by completing the squares in the x and y terms.

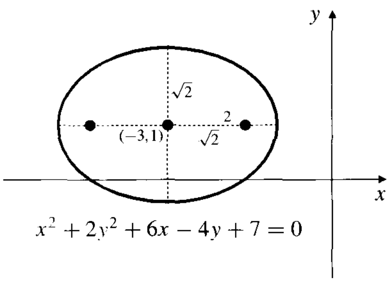


Figure 8.14

Example 4 Describe the curve with equation $x^2+2y^2+6x-4y+7=0$.

Solution We complete the squares in the x and y terms, and rewrite the equation in the form

$$x^2+6x+9+2(y^2-2y+1)=9+2-7=4$$
$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{2}=1.$$

Therefore, it represents an ellipse with centre at $(-3, 1)$, semi-major axis $a=2$, and semi-minor axis $b=\sqrt{2}$. Since $c=\sqrt{a^2-b^2}=\sqrt{2}$, the foci are $(-3\pm\sqrt{2}, 1)$. See Figure 8.14.

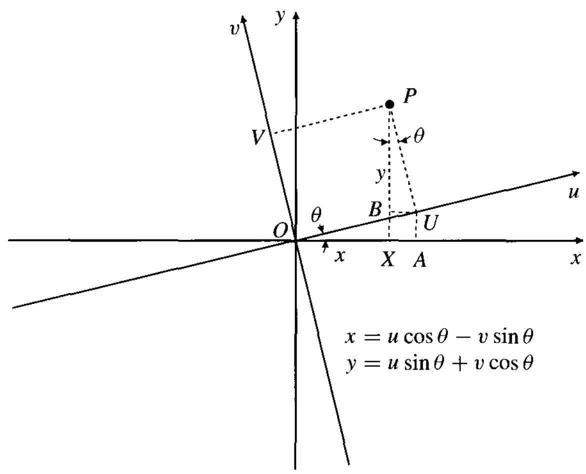


Figure 8.15 Rotation of axes

If $B\neq 0$, the equation has an xy term, and it cannot represent a circle. To see what it does represent, we can rotate the coordinate axes to produce an equation with no xy term. Let new coordinate axes (a u-axis and a v-axis) have the same origin but be rotated an angle θ from the x- and y-axes, respectively. (See Figure 8.15.) If point P has coordinates (x, y) with respect to the old axes and coordinates (u, v) with respect to the new axes, then an analysis of triangles in the figure shows that

$$x=OA-XA=OU\cos\theta-OV\sin\theta=u\cos\theta-v\sin\theta,$$
$$y=XB+BP=OU\sin\theta+OV\cos\theta=u\sin\theta+v\cos\theta.$$

Substituting these expressions into the equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \qquad (A^2+B^2+C^2>0)$$

leads to a new equation,

$$A'u^2+B'uv+C'v^2+D'u+E'v+F=0,$$

where

BEWARE
A lengthy calculation is needed here. The details have been omitted.

$$\begin{aligned}
A' &= \frac{1}{2}\Big(A(1+\cos 2\theta)+B\sin 2\theta+C(1-\cos 2\theta)\Big)\\
B' &= (C-A)\sin 2\theta+B\cos 2\theta\\
C' &= \frac{1}{2}\Big(A(1-\cos 2\theta)-B\sin 2\theta+C(1+\cos 2\theta)\Big)\\
D' &= D\cos\theta+E\sin\theta\\
E' &= -D\sin\theta+E\cos\theta.
\end{aligned}$$

Note that F remains unchanged. If we choose θ so that

$$\tan 2\theta = \frac{B}{A-C}, \qquad \text{or} \qquad \theta = \frac{\pi}{4} \text{ if } A = C,\ B \neq 0,$$

then $B' = 0$, and the new equation can then be analyzed as described previously.

Example 5 Identify the curve with equation $xy = 1$.

Solution The reader is likely well aware that the given equation represents a rectangular hyperbola with the coordinate axes as asymptotes. Since the given equation involves $A = C = D = E = 0$ and $B = 1$, it is appropriate to rotate the axes through angle $\pi/4$ so that

$$x = \frac{1}{\sqrt{2}}(u - v), \qquad y = \frac{1}{\sqrt{2}}(u + v).$$

The transformed equation is $u^2 - v^2 = 2$, which is, as suspected, a rectangular hyperbola with vertices at $u = \pm\sqrt{2}$, $v = 0$, foci at $u = \pm 2$, $v = 0$, and asymptotes $u = \pm v$. Hence, $xy = 1$ represents a rectangular hyperbola with coordinate axes as asymptotes, vertices at $(1, 1)$ and $(-1, -1)$, and foci at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. ■

Example 6 Show that the curve $2x^2 + xy + y^2 = 2$ is an ellipse, and find the lengths of its semi-major and semi-minor axes.

Solution Here, $A = 2$, $B = C = 1$, $D = E = 0$, and $F = -2$. We rotate the axes through angle θ where $\tan 2\theta = B/(A - C) = 1$. Thus $B' = 0$, $2\theta = \pi/4$, and $\sin 2\theta = \cos 2\theta = 1/\sqrt{2}$. We have

$$A' = \frac{1}{2}\left[2\left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)\right] = \frac{3 + \sqrt{2}}{2}$$

$$C' = \frac{1}{2}\left[2\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right)\right] = \frac{3 - \sqrt{2}}{2}.$$

The transformed equation is $(3 + \sqrt{2})u^2 + (3 - \sqrt{2})v^2 = 4$, which represents an ellipse with semi-major axis $2/\sqrt{3 - \sqrt{2}}$ and semi-minor axis $2/\sqrt{3 + \sqrt{2}}$. (We will discover another way to do a question like this in Section 13.3.) ■

Exercises 8.1

Find equations of the conics specified in Exercises 1–6.

1. ellipse with foci at $(0, \pm 2)$ and semi-major axis 3.
2. ellipse with foci at $(0, 1)$ and $(4, 1)$ and eccentricity $1/2$.
3. parabola with focus at $(2, 3)$ and vertex at $(2, 4)$.
4. parabola passing through the origin and having focus at $(0, -1)$ and axis along $y = -1$.
5. hyperbola with foci at $(0, \pm 2)$ and semi-transverse axis 1.
6. hyperbola with foci at $(\pm 5, 1)$ and asymptotes $x = \pm(y - 1)$.

In Exercises 7–15, identify and sketch the set of points in the plane satisfying the given equation. Specify the asymptotes of any hyperbolas.

7. $x^2 + y^2 + 2x = -1$
8. $x^2 + 4y^2 - 4y = 0$
9. $4x^2 + y^2 - 4y = 0$
10. $4x^2 - y^2 - 4y = 0$
11. $x^2 + 2x - y = 3$
12. $x + 2y + 2y^2 = 1$
13. $x^2 - 2y^2 + 3x + 4y = 2$

14. $9x^2 + 4y^2 - 18x + 8y = -13$

15. $9x^2 + 4y^2 - 18x + 8y = 23$

16. Identify and sketch the curve that is the graph of the equation $(x - y)^2 - (x + y)^2 = 1$.

* **17.** Light rays in the xy-plane coming from the point $(3, 4)$ reflect in a parabola so that they form a beam parallel to the x-axis. The parabola passes through the origin. Find its equation. (There are two possible answers.)

18. Light rays in the xy-plane coming from the origin are reflected by an ellipse so that they converge at the point $(3, 0)$. Find all possible equations for the ellipse.

In Exercises 19–22, identify the conic and find its centre, principal axes, foci, and eccentricity. Specify the asymptotes of any hyperbolas.

19. $xy + x - y = 2$

* **20.** $x^2 + 2xy + y^2 = 4x - 4y + 4$

* **21.** $8x^2 + 12xy + 17y^2 = 20$

* **22.** $x^2 - 4xy + 4y^2 + 2x + y = 0$

23. The *focus-directrix definition of a conic* defines a conic as a set of points P in the plane that satisfy the condition

$$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to } D} = \varepsilon,$$

where F is a fixed point, D a fixed straight line, and ε a fixed positive number. The conic is an ellipse, a parabola, or a hyperbola according to whether $\varepsilon < 1$, $\varepsilon = 1$, or $\varepsilon > 1$. Find the equation of the conic if F is the origin and D is the line $x = -p$.

Another parameter associated with conics is the **semi-latus rectum**, usually denoted ℓ. For a circle it is equal to the radius. For other conics it is half the length of the chord through a focus and perpendicular to the axis (for a parabola), the major axis (for an ellipse), or the transverse axis (for a hyperbola). That chord is called the **latus rectum** of the conic.

24. Show that the semi-latus rectum of the parabola is twice the distance from the vertex to the focus.

25. Show that the semi-latus rectum for an ellipse with semi-major axis a and semi-minor axis b is $\ell = b^2/a$.

26. Show that the formula in the above exercise also gives the semi-latus rectum of a hyperbola with semi-transverse axis a and semi-conjugate axis b.

* **27.** Suppose a plane intersects a right-circular cone in an ellipse and that two spheres (one on each side of the plane) are inscribed between the cone and the plane so that each is tangent to the cone around a circle and is also tangent to the plane at a point. Show that the points where these two spheres touch the plane are the foci of the ellipse. *Hint:* all tangent lines drawn to a sphere from a given point outside the sphere are equal in length. The distance between the two circles in which the spheres intersect the cone, measured along generators of the cone (i.e., straight lines lying on the cone), is the same for all generators.

* **28.** State and prove a result analogous to that in the above exercise but pertaining to a hyperbola.

* **29.** Suppose a plane intersects a right-circular cone in a parabola with vertex at V. Suppose that a sphere is inscribed between the cone and the plane as in the previous exercises and is tangent to the plane of the parabola at point F. Show that the chord to the parabola through F which is perpendicular to FV has length equal to that of the latus rectum of the parabola. Therefore, F is the focus of the parabola.

8.2 Parametric Curves

Suppose that an object moves around in the xy-plane so that the coordinates of its position at any time t are continuous functions of the variable t:

$$x = f(t), \qquad y = g(t).$$

The path followed by the object is a curve $\mathcal{C}$ in the plane that is specified by the two equations above. We call these equations *parametric equations* of $\mathcal{C}$. A curve specified by a particular pair of parametric equations is called a *parametric curve*.

DEFINITION 4

Parametric curves

A **parametric curve** $\mathcal{C}$ in the plane consists of an ordered pair (f, g) of continuous functions each defined on the same interval I. The equations

$$x = f(t), \qquad y = g(t), \qquad \text{for } t \text{ in } I,$$

are called **the parametric equations** of the curve $\mathcal{C}$. The independent variable t is called the **parameter**.

Note that the parametric curve $\mathcal{C}$ was *not* defined as a set of points in the plane, but rather as the ordered pair of functions whose range is that set of points. Different pairs of functions can give the same set of points in the plane, but we may still want to regard them as different parametric curves. Nevertheless, we will often refer to the set of points (the path traced out by (x, y) as t traverses I) as the curve $\mathcal{C}$. The axis (real line) of the parameter t is distinct from the coordinate axes of the plane of the curve. (See Figure 8.16.) We will usually denote the parameter by t; in many applications the parameter represents time, but this need not always be the case. Because f and g are assumed to be continuous, the curve $x = f(t)$, $y = g(t)$ has no breaks in it. A parametric curve has a *direction* (indicated, say, by arrowheads), namely, the direction corresponding to increasing values of the parameter t, as shown in Figure 8.16.

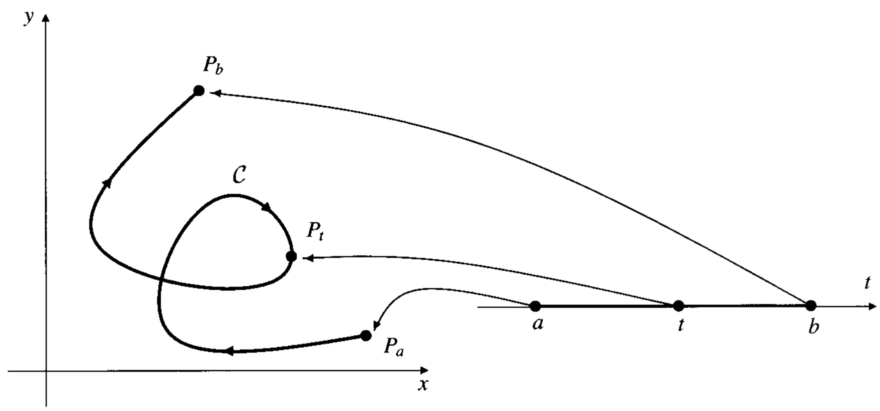


Figure 8.16 A parametric curve

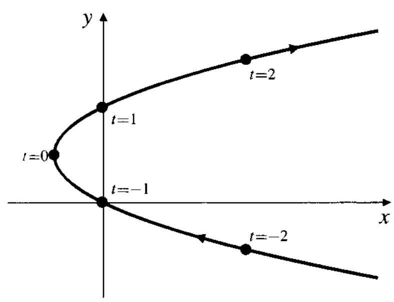


Figure 8.17 The parabola defined parametrically by $x = t^2 - 1$, $y = t + 1$, $(-\infty < t < \infty)$

Example 1 Sketch and identify the parametric curve

$$x = t^2 - 1, \qquad y = t + 1 \qquad (-\infty < t < \infty).$$

Solution We could construct a table of values of x and y for various values of t, thus getting the coordinates of a number of points on a curve. However, for this example it is easier to *eliminate the parameter* from the pair of parametric equations, thus producing a single equation in x and y whose graph is the desired curve:

$$t = y - 1, \qquad x = t^2 - 1 = (y - 1)^2 - 1 = y^2 - 2y.$$

All points on the curve lie on the parabola $x = y^2 - 2y$. Since $y \to \pm\infty$ as $t \to \pm\infty$, the parametric curve is the whole parabola. (See Figure 8.17.)

Although the curve in Example 1 is more easily identified when the parameter is eliminated, there is a loss of information in going to the nonparametric form. Specifically, we lose the sense of the curve as the path of a moving point and hence also the direction of the curve. If the t in the parametric form denotes the time at which an object is at the point (x, y), the nonparametric equation $x = y^2 - 2y$ no longer tells us where the object is at any particular time t.

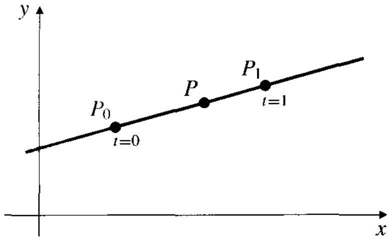


Figure 8.18

Example 2 **(Parametric equations of a straight line)** The straight line passing through the two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ (see Figure 8.18) has parametric equations

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \end{cases} \qquad (-\infty < t < \infty).$$

To see that these equations represent a straight line, note that

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \text{constant} \qquad (\text{assuming } x_1 \neq x_0).$$

The point $P = (x, y)$ is at position P_0 when $t = 0$ and at P_1 when $t = 1$. If $t = 1/2$, then P is the midpoint between P_0 and P_1. Note that the line segment from P_0 to P_1 corresponds to values of t between 0 and 1.

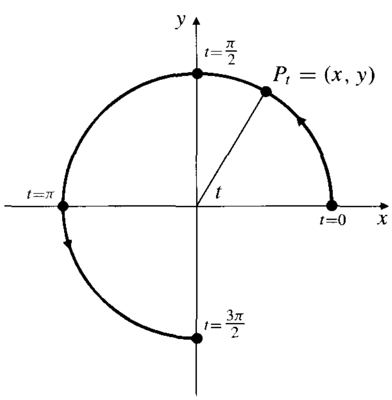


Figure 8.19

Example 3 **(An arc of a circle)** Sketch and identify the curve $x = 3\cos t$, $y = 3\sin t$, $(0 \le t \le 3\pi/2)$.

Solution Since $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$, all points on the curve lie on the circle $x^2 + y^2 = 9$. As t increases from 0 through $\pi/2$ and π to $3\pi/2$, the point (x, y) moves from $(3, 0)$ through $(0, 3)$ and $(-3, 0)$ to $(0, -3)$. The parametric curve is three-quarters of the circle. See Figure 8.19. The parameter t has geometric significance in this example. If P_t is the point on the curve corresponding to parameter value t, then t is the angle at the centre of the circle corresponding to the arc from the initial point to P_t.

Example 4 **(Parametric equations of an ellipse)** Sketch and identify the curve $x = a\cos t$, $y = b\sin t$, $(0 \le t \le 2\pi)$, where $a > b > 0$.

Solution Observe that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1.$$

Therefore, the curve is all or part of an ellipse with major axis from $(-a, 0)$ to $(a, 0)$ and minor axis from $(0, -b)$ to $(0, b)$. As t increases from 0 to 2π, the point (x, y) moves counterclockwise around the ellipse starting from $(a, 0)$ and returning to the same point. Thus the curve is the whole ellipse.

Figure 8.20(a) shows how the parameter t can be interpreted as an angle and how the points on the ellipse can be obtained using circles of radii a and b. Since the curve starts and ends at the same point, it is called a **closed curve**.

Example 5 Sketch the parametric curve

$$x = t^3 - 3t, \qquad y = t^2 \qquad (-2 \le t \le 2).$$

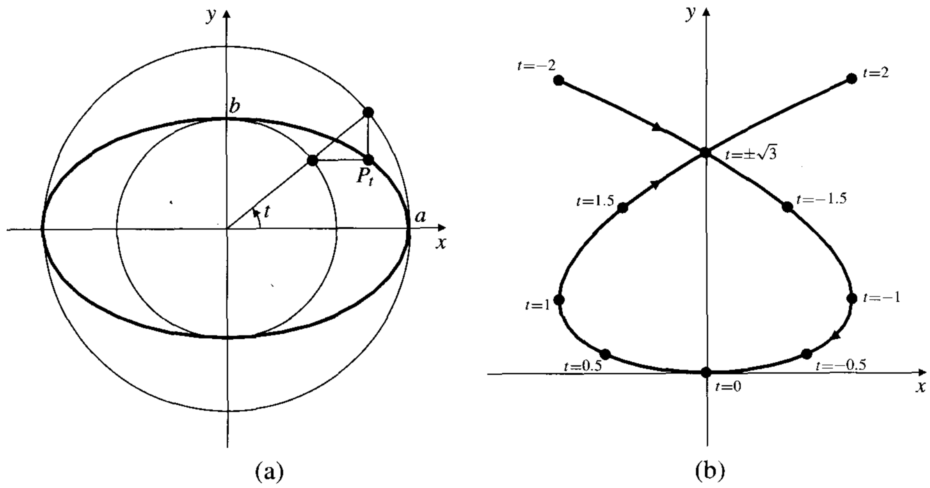


Figure 8.20

(a) An ellipse parametrized in terms of an angle and constructed with the help of two circles

(b) A self-intersecting parametric curve

Solution We could eliminate the parameter and obtain

$$x^2 = t^2(t^2 - 3)^2 = y(y - 3)^2,$$

but this doesn't help much since we do not recognize this curve from its Cartesian equation. Instead, let us calculate the coordinates of some points:

Table 2. Coordinates of some points on the curve of Example 5

t	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	-2	$\frac{9}{8}$	2	$\frac{11}{8}$	0	$-\frac{11}{8}$	-2	$-\frac{9}{8}$	2
y	4	$\frac{9}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4

Note that the curve is symmetric about the y-axis because x is an odd function of t and y is an even function of t. (At t and $-t$, x has opposite values but y has the same value.)

The curve intersects itself on the y-axis. (See Figure 8.20(b).) To find this self-intersection, set $x = 0$:

$$0 = x = t^3 - 3t = t(t - \sqrt{3})(t + \sqrt{3}).$$

For $t = 0$ the curve is at $(0, 0)$, but for $t = \pm\sqrt{3}$ the curve is at $(0, 3)$. The self-intersection occurs because the curve passes through the same point for two different values of the parameter. ■

Remark Here is how to get Maple to plot the parametric curve in the example above. Note the square brackets enclosing the two functions $t^3 - 3t$ and t^2, and the parameter interval, followed by the ranges of x and y for the plot.

```
> plot([t^3-3*t, t^2, t=-2..2], x=-3..3, y=-1..5);
```

General Plane Curves and Parametrizations

According to Definition 4, a parametric curve always involves a particular set of parametric equations; it is not just a set of points in the plane. When we are interested in considering a curve solely as a set of points (a *geometric object*), we need not be concerned with any particular pair of parametric equations representing that curve. In this case we call the curve simply a *plane curve*.

DEFINITION 5

Plane curves

A **plane curve** is a set of points (x, y) in the plane such that $x = f(t)$ and $y = g(t)$ for some t in an interval I, where f and g are continuous functions defined on I. Any such interval I and function pair (f, g) that generate the points of $\mathcal{C}$ is called a **parametrization** of $\mathcal{C}$.

Since a plane curve does not involve any specific parametrization, it has no specific direction.

Example 6 The circle $x^2 + y^2 = 1$ is a plane curve. Each of the following is a possible parametrization of C:

(i) $x = \cos t,\ y = \sin t, \qquad (0 \le t \le 2\pi)$,

(ii) $x = \sin s^2,\ y = \cos s^2, \qquad (0 \le s \le \sqrt{2\pi})$,

(iii) $x = \cos(\pi u + 1), y = \sin(\pi u + 1), \qquad (-1 \le u \le 1)$,

(iv) $x = 1 - t^2, y = t\sqrt{2 - t^2}, \qquad (-\sqrt{2} \le t \le \sqrt{2})$.

To verify that any of these represents the circle, substitute the appropriate functions for x and y in the expression $x^2 + y^2$, and show that the result simplifies to the value 1. This shows that the parametric curve lies on the circle. Then examine the ranges of x and y as the parameter varies over its domain. For example, for (iv) we have

$$x^2 + y^2 = (1 - t^2)^2 + (t\sqrt{2 - t^2})^2 = 1 - 2t^2 + t^4 + 2t^2 - t^4 = 1,$$

and (x, y) moves from $(-1, 0)$ through $(0, -1)$ to $(1, 0)$ as t increases from $-\sqrt{2}$ through -1 to 0, and then continues on through $(0, 1)$ back to $(-1, 0)$ as t continues to increase from 0 through 1 to $\sqrt{2}$.

There are, of course, infinitely many other possible parametrizations of this curve.

■

Example 7 If f is a continuous function on an interval I, then the graph of f is a plane curve. One obvious parametrization of this curve is

$$x = t, \qquad y = f(t), \qquad (t \text{ in } I).$$

■

Some Interesting Plane Curves

We complete this section by parametrizing two curves that arise in the physical world.

The brachistochrone and tautochrone problems
Suppose a wire is bent into a curve from point A to a lower point B and a bead can slide without friction along the wire. If the bead is released at A, it will fall toward B. What curve should be used to minimize the time it takes to fall from A to B? This problem, known as the *brachistochrone* (Greek for "shortest time") problem, has as its solution part of an upside down arch of a cycloid. Moreover, it takes the same amount of time for the bead to slide from any point on the curve to the lowest point B, making the cycloid the solution of the *tautochrone* ("equal time") problem as well. We will examine these matters further in the Challenging Exercises at the end of Chapter 11.

Example 8 **(A cycloid)** If a circle rolls without slipping along a straight line, find the path followed by a point fixed on the circle. This path is called a **cycloid**.

Solution Suppose that the line on which the circle rolls is the x-axis, that the circle has radius a and lies above the line, and that the point whose motion we follow is originally at the origin O. See Figure 8.21. After the circle has rolled through an angle t, it is tangent to the line at T, and the point whose path we are trying to find has moved to position P, as shown in the figure. Since no slipping occurs,

$$\text{segment } OT = \text{arc } PT = at.$$

Let PQ be perpendicular to TC, as shown in the figure. If P has coordinates (x, y), then

$$x = OT - PQ = at - a\sin(\pi - t) = at - a\sin t,$$
$$y = TC + CQ = a + a\cos(\pi - t) = a - a\cos t.$$

The parametric equations of the cycloid are therefore

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

Observe that the cycloid has a cusp at the points where it returns to the x-axis, that is, at points corresponding to $t = 2n\pi$, where n is an integer. Even though the functions x and y are everywhere differentiable functions of t, the curve is not smooth everywhere. We shall consider such matters in the next section. ∎

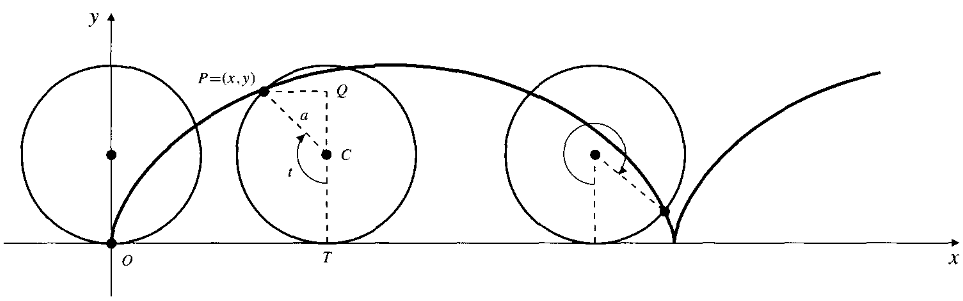


Figure 8.21 Each arch of the cycloid is traced out by P as the wheel rolls through one complete revolution

Example 9 **(An involute of a circle)** A string is wound around a fixed circle. One end is unwound in such a way that the part of the string not lying on the circle is extended in a straight line. The curve followed by this free end of the string is called an **involute** of the circle. (The involute of any curve is the path traced out by the end of the curve as the curve is straightened out beginning at that end.)

Suppose the circle has equation $x^2 + y^2 = a^2$, and suppose the end of the string being unwound starts at the point $A = (a, 0)$. At some subsequent time during the unwinding let P be the position of the end of the string, and let T be the point where the string leaves the circle. The line PT must be tangent to the circle at T.

We parametrize the path of P in terms of the angle AOT, which we denote by t. Let points R on OA and S on TR be as shown in Figure 8.22. TR is perpendicular to OA and to PS. Note that

$$OR = OT\cos t = a\cos t, \qquad RT = OT\sin t = a\sin t.$$

Since angle OTP is $90°$, we have angle $STP = t$. Since $PT = \text{arc}\,AT = at$ (because the string does not stretch or slip on the circle), we have

$$SP = TP\sin t = at\sin t, \qquad ST = TP\cos t = at\cos t.$$

If P has coordinates (x, y), then $x = OR + SP$, and $y = RT - ST$:

$$x = a\cos t + at\sin t, \qquad y = a\sin t - at\cos t, \qquad (t \ge 0).$$

These are parametric equations of the involute. ■

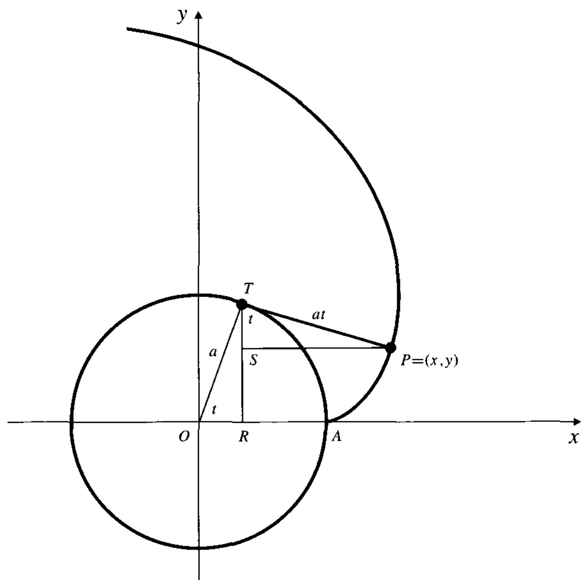


Figure 8.22 An involute of a circle

Exercises 8.2

In Exercises 1–10, sketch the given parametric curve, showing its direction with an arrow. Eliminate the parameter to give a Cartesian equation in x and y whose graph contains the parametric curve.

1. $x = 1 + 2t,\ y = t^2,\ (-\infty < t < \infty)$
2. $x = 2 - t,\ y = t + 1,\ (0 \le t < \infty)$
3. $x = \dfrac{1}{t},\ y = t - 1,\ (0 < t < 4)$
4. $x = \dfrac{1}{1+t^2},\ y = \dfrac{t}{1+t^2},\ (-\infty < t < \infty)$
5. $x = 3\sin 2t,\ y = 3\cos 2t,\ \left(0 \le t \le \dfrac{\pi}{3}\right)$
6. $x = a\sec t,\ y = b\tan t,\ \left(-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\right)$
7. $x = 3\sin \pi t,\ y = 4\cos \pi t,\ (-1 \le t \le 1)$
8. $x = \cos\sin s,\ y = \sin\sin s,\ (-\infty < s < \infty)$
9. $x = \cos^3 t,\ y = \sin^3 t,\ (0 \le t \le 2\pi)$
10. $x = 1 - \sqrt{4 - t^2},\ y = 2 + t,\ (-2 \le t \le 2)$
11. Describe the parametric curve $x = \cosh t$, $y = \sinh t$, and find its Cartesian equation.

12. Describe the parametric curve $x = 2 - 3\cosh t$, $y = -1 + 2\sinh t$.

13. Describe the curve $x = t\,\cos t$, $y = t\,\sin t$, $(0 \le t \le 4\pi)$.

14. Show that each of the following sets of parametric equations represents a different arc of the parabola with equation $2(x + y) = 1 + (x - y)^2$.

(i) $x = \cos^4 t,\ \ y = \sin^4 t$

(ii) $x = \sec^4 t,\ \ y = \tan^4 t$

the point $(\lambda, 0)$. Does the parametrization fail to give any point on the circle?

17. A circle of radius a is centred at the origin O. T is a point on the circle such that OT makes angle t with the positive x-axis. The tangent to the circle at T meets the x-axis at X. The point $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through T. Find, in terms of the parameter t, parametric equations for the curve $\mathcal{C}$ traced out by P as T moves around the circle. Also, eliminate t and find an equation for $\mathcal{C}$ in x and y. Sketch $\mathcal{C}$.

18. Repeat Exercise 17 with the following modification. OT meets a second circle of radius b centred at O at the point Y. $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through Y.

$*$ **19.** **(The folium of Descartes)** Eliminate the parameter from the parametric equations

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3} \qquad (t \neq -1),$$

and hence find an ordinary equation in x and y for this curve. The parameter t can be interpreted as the slope of the line joining the general point (x, y) to the origin. Sketch the curve and show that the line $x + y = -1$ is an asymptote.

$*$ **20.** **(A prolate cycloid)** A railroad wheel has a flange extending below the level of the track on which the wheel rolls. If the radius of the wheel is a and that of the flange is $b > a$, find parametric equations of the path of a point P at the circumference of the flange as the wheel rolls along the track. (Note that for a portion of each revolution of the wheel, P is moving backward.) Try to sketch the graph of this prolate cycloid.

$*$ **21.** **(Hypocycloids)** If a circle of radius b rolls, without slipping, around the inside of a fixed circle of radius $a > b$, a point on the circumference of the rolling circle traces a curve called a hypocycloid. If the fixed circle is centred at the origin and the point tracing the curve starts at $(a, 0)$, show that the hypocycloid has parametric equations

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right),$$
$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right),$$

where t is the angle between the positive x-axis and the line from the origin to the point at which the rolling circle curve is called a hypocycloid of four cusps or an **astroid**. (See Figure 8.23.) It has Cartesian equation $x^{2/3} + y^{2/3} = 4^{2/3}$.

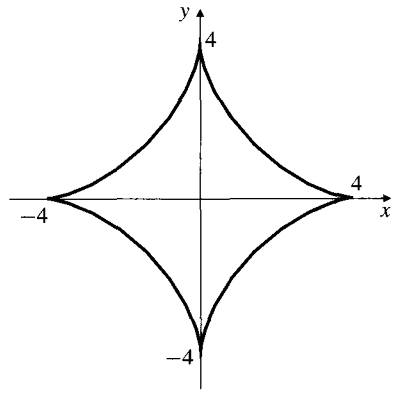


Figure 8.23 The astroid $x^{2/3} + y^{2/3} = 4^{2/3}$

Hypocycloids resemble the curves produced by a popular children's toy called Spirograph, but Spirograph curves result from following a point inside the disc of the rolling circle rather than on its circumference, and they therefore do not have sharp cusps.

$*$ **22.** **(The witch of Agnesi)**

(a) Show that the curve traced out by the point P constructed from a circle as shown in Figure 8.24 has parametric equations $x = \tan t,\ y = \cos^2 t$ in terms of the angle t shown. (*Hint:* you will need to make extensive use of similar triangles.)

(b) Use a trigonometric identity to eliminate t from the parametric equations, and hence find an ordinary Cartesian equation for the curve.

This curve is named for the Italian mathematician Maria Agnesi (1718-1799), one of the foremost women scholars of her century and author of an important calculus text. The term *witch* is due to a mistranslation of the Italian word *versiera* ("turning curve"), which she used to describe the curve. The word is similar to *avversiera* ("wife of the devil" or "witch").

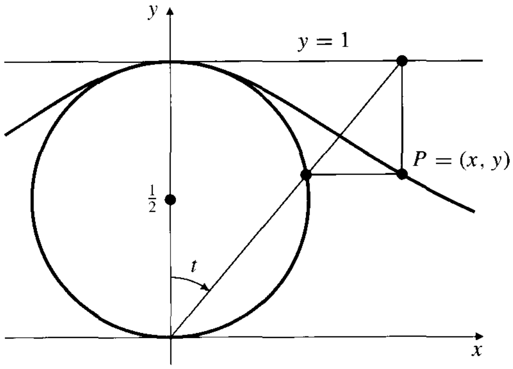


Figure 8.24 The witch of Agnesi

In Exercises 23–26, obtain a graph of the curve $x = \sin(mt)$, $y = \sin(nt)$ for the given values of m and n. Such curves are called **Lissajous figures.** They arise in the analysis of electrical signals using an oscilloscope. A signal of fixed but unknown frequency is applied to the vertical input, and a control signal is applied to the horizontal input. The horizontal frequency is varied until a stable Lissajous figure is observed. The (known) frequency of the control signal and the shape of the figure then determine the unknown frequency.

23. $m = 1, \quad n = 2$

24. $m = 1, \quad n = 3$

25. $m = 2, \quad n = 3$

26. $m = 2, \quad n = 5$

27. **(Epicycloids)** Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t - \frac{1}{n}\cos(nt)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin(nt)$$

for various integer and fractional values of $n \geq 3$. Can you formulate any principles governing the behaviour of such curves?

28. **(More hypocycloids)** Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t + \frac{1}{n}\cos((n-1)t)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin((n-1)t)$$

for various integer and fractional values of $n \geq 3$. Can you formulate any principles governing the behaviour of these curves?

8.3 Smooth Parametric Curves and Their Slopes

We say that a plane curve is *smooth* if it has a tangent line at each point P and this tangent turns in a continuous way as P moves along the curve. (That is, the angle between the tangent line at P and some fixed line, the x-axis say, is a continuous function of the position of P.)

If the curve $\mathcal{C}$ is the graph of function f, then $\mathcal{C}$ is certainly smooth on any interval where the derivative $f'(x)$ exists and is a continuous function of x. It may also be smooth on intervals containing isolated singular points; for example, the curve $y = x^{1/3}$ is smooth everywhere even though dy/dx does not exist at $x = 0$.

For parametric curves $x = f(t)$, $y = g(t)$, the situation is more complicated. Even if f and g have continuous derivatives everywhere, such curves may fail to be smooth at certain points, specifically points where $f'(t) = g'(t) = 0$.

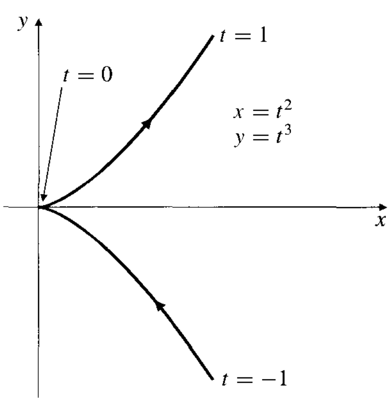


Figure 8.25 This curve is not smooth at the origin but has a cusp there

Example 1 Consider the parametric curve $x = f(t) = t^2$, $y = g(t) = t^3$. Even though $f'(t) = 2t$ and $g'(t) = 3t^2$ are continuous for all t, the curve is not smooth at $t = 0$. (See Figure 8.25.) Observe that both f' and g' vanish at $t = 0$: $f'(0) = g'(0) = 0$. If we regard the parametric equations as specifying the position at time t of a moving point P, then the horizontal velocity is $f'(t)$ and the vertical velocity is $g'(t)$. Both velocities are 0 at $t = 0$, so P has come to a stop at that instant. When it starts moving again, it need not move in the direction it was going before it stopped. The cycloid of Example 8 of Section 8.2 is another example where a parametric curve is not smooth at points where dx/dt and dy/dt both vanish. ■

The Slope of a Parametric Curve

The following theorem confirms that a parametric curve is smooth at points where the derivatives of its coordinate functions are continuous and not both zero.

THEOREM 1

Let $\mathcal{C}$ be the parametric curve $x = f(t)$, $y = g(t)$, where $f'(t)$ and $g'(t)$ are continuous on an interval I. If $f'(t) \neq 0$ on I, then $\mathcal{C}$ is smooth and has at each t a tangent line with slope

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

If $g'(t) \neq 0$ on I, then $\mathcal{C}$ is smooth and has at each t a normal line with slope

$$-\frac{dx}{dy} = -\frac{f'(t)}{g'(t)}.$$

Thus, $\mathcal{C}$ is smooth except possibly at points where $f'(t)$ and $g'(t)$ are both 0.

PROOF If $f'(t) \neq 0$ on I, then f is either increasing or decreasing on I and so is one-to-one and invertible. The part of $\mathcal{C}$ corresponding to values of t in I has ordinary equation $y = g\big(f^{-1}(x)\big)$ and hence slope

$$\frac{dy}{dx} = g'\big(f^{-1}(x)\big)\,\frac{d}{dx}f^{-1}(x) = \frac{g'\big(f^{-1}(x)\big)}{f'\big(f^{-1}(x)\big)} = \frac{g'(t)}{f'(t)}.$$

We have used here the formula

$$\frac{d}{dx}\,f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

for the derivative of an inverse function obtained in Section 3.1. This slope is a continuous function of t, so the tangent to $\mathcal{C}$ turns continuously for t in I. The proof for $g'(t) \neq 0$ is similar. In this case the slope of the normal is a continuous function of t, so the normal turns continuously. Therefore so does the tangent.

If f' and g' are continuous, and both vanish at some point t_0, then the curve $x = f(t)$, $y = g(t)$ *may or may not* be smooth around t_0. Example 1 was an example of a curve that was not smooth at such a point.

Example 2 The curve with parametrization $x = t^3$, $y = t^6$ is just the parabola $y = x^2$, so it is smooth everywhere, although $dx/dt = 3t^2$ and $dy/dt = 6t^5$ both vanish at $t = 0$.

Tangents and normals to parametric curves

If f' and g' are continuous and not both 0 at t_0, then the parametric equations

$$\begin{cases} x = f(t_0) + f'(t_0)(t - t_0) \\ y = g(t_0) + g'(t_0)(t - t_0) \end{cases} \qquad (-\infty < t < \infty)$$

represent the tangent line to the parametric curve $x = f(t)$, $y = g(t)$ at the point $\big(f(t_0), g(t_0)\big)$. The normal line there has parametric equations

$$\begin{cases} x = f(t_0) + g'(t_0)(t - t_0) \\ y = g(t_0) - f'(t_0)(t - t_0) \end{cases} \qquad (-\infty < t < \infty).$$

Both lines pass through $\big(f(t_0), g(t_0)\big)$ when $t = t_0$.

Example 3 Find equations of the tangent and normal lines to the parametric curve $x = t^2 - t,\ y = t^2 + t$ at the point where $t = 2$.

Solution At $t = 2$ we have $x = 2$, $y = 6$ and

$$\frac{dx}{dt} = 2t - 1 = 3, \qquad \frac{dy}{dt} = 2t + 1 = 5.$$

Hence, the tangent and the normal lines have parametric equations

$$\text{Tangent:} \quad \begin{cases} x = 2 + 3(t - 2) = 3t - 4 \\ y = 6 + 5(t - 2) = 5t - 4 \end{cases}$$

$$\text{Normal:} \quad \begin{cases} x = 2 + 5(t - 2) = 5t - 8 \\ y = 6 - 3(t - 2) = -3t + 12 \end{cases}$$

■

The concavity of a parametric curve can be determined using the second derivatives of the parametric equations. The procedure is just to calculate d^2y/dx^2 using the chain rule:

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{g'(t)}{f'(t)} &= \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\frac{dt}{dx} \\ &= \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^2}\,\frac{1}{f'(t)}. \end{aligned}$$

Concavity of a parametric curve

On an interval where $f'(t) \neq 0$, the parametric curve $x = f(t)$, $y = g(t)$ has concavity determined by

$$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}.$$

Sketching Parametric Curves

As in the case of graphs of functions, derivatives provide useful information about the shape of a parametric curve. At points where $dy/dt = 0$ but $dx/dt \neq 0$, the tangent is horizontal; at points where $dx/dt = 0$ but $dy/dt \neq 0$, the tangent is vertical. For points where $dx/dt = dy/dt = 0$, anything can happen; it is wise to calculate left- and right-hand limits of the slope dy/dx as the parameter t approaches one of these points. Concavity can be determined using the formula obtained above. We illustrate these ideas by reconsidering a parametric curve encountered in the previous section.

Example 4 Use slope and concavity information to sketch the graph of the parametric curve

$$x = f(t) = t^3 - 3t, \qquad y = g(t) = t^2, \qquad (-2 \le t \le 2)$$

previously encountered in Example 5 of Section 8.2.

Solution We have

$$f'(t) = 3(t^2 - 1) = 3(t - 1)(t + 1), \qquad g'(t) = 2t.$$

The curve has a horizontal tangent at $t = 0$, that is, at $(0, 0)$, and vertical tangents at $t = \pm 1$, that is, at $(2, 1)$ and $(-2, 1)$. Directional information for the curve between these points is summarized in the following chart.

t	-2		-1		0		1		2
$f'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$g'(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	
x		$\rightarrow$	$\cdot$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\cdot$	$\rightarrow$	
y		$\downarrow$	$\downarrow$	$\downarrow$	$\cdot$	$\uparrow$	$\uparrow$	$\uparrow$	
curve		$\searrow$	$\downarrow$	$\swarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$	

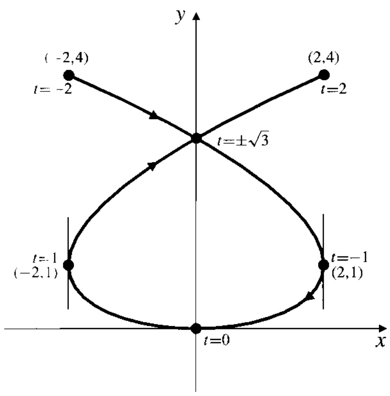


Figure 8.26

For concavity we calculate the second derivative d^2y/dx^2 by the formula obtained above. Since $f''(t) = 6t$ and $g''(t) = 2$, we have

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3} \\ &= \frac{3(t^2 - 1)(2) - 2t(6t)}{[3(t^2 - 1)]^3} = -\frac{2}{9}\frac{t^2 + 1}{(t^2 - 1)^3}, \end{aligned}$$

which is never zero but which fails to be defined at $t = \pm 1$. Evidently the curve is concave upward for $-1 < t < 1$ and concave downward elsewhere. The curve is sketched in Figure 8.26. ■

Exercises 8.3

In Exercises 1–8, find the coordinates of the points at which the given parametric curve has (a) a horizontal tangent and (b) a vertical tangent.

1. $x = t^2 + 1, \ y = 2t - 4$ **2.** $x = t^2 - 2t, \ y = t^2 + 2t$

3. $x = t^2 - 2t, \ y = t^3 - 12t$

4. $x = t^3 - 3t, \ y = 2t^3 + 3t^2$

5. $x = te^{-t^2/2}, \ y = e^{-t^2}$

6. $x = \sin t, \ y = \sin t - t\cos t$

7. $x = \sin 2t, \ y = \sin t$ **8.** $x = \dfrac{3t}{1 + t^3}, \ y = \dfrac{3t^2}{1 + t^3}$

Find the slopes of the curves in Exercises 9–12 at the points indicated.

9. $x = t^3 + t,\ y = 1 - t^3,$ at $t = 1$

10. $x = t^4 - t^2,\ y = t^3 + 2t,$ at $t = -1$

11. $x = \cos 2t,\ y = \sin t,$ at $t = \pi/6$

12. $x = e^{2t},\ y = te^{2t},$ at $t = -2$

Find parametric equations of the tangents to the curves in Exercises 13–14 at the indicated points.

13. $x = t^3 - 2t,\ y = t + t^3,$ at $t = 1$

14. $x = t - \cos t,\ y = 1 - \sin t,$ at $t = \pi/4$

15. Show that the curve $x = t^3 - t$, $y = t^2$ has two different tangent lines at the point $(0, 1)$ and find their slopes.

16. Find the slopes of two lines that are tangent to $x = \sin t$, $y = \sin 2t$ at the origin.

Where, if anywhere, do the curves in Exercises 17–20 fail to be smooth?

17. $x = t^3,\ y = t^2$

18. $x = (t - 1)^4,\ y = (t - 1)^3$

19. $x = t \sin t,\ y = t^3$

20. $x = t^3,\ y = t - \sin t$

In Exercises 21–25, sketch the graphs of the given parametric curves, making use of information from the first two derivatives. Unless otherwise stated, the parameter interval for each curve is the whole real line.

21. $x = t^2 - 2t,\ y = t^2 - 4t$

22. $x = t^3,\ y = 3t^2 - 1$

23. $x = t^3 - 3t,\ y = \dfrac{2}{1 + t^2}$

24. $x = t^3 - 3t - 2,\ y = t^2 - t - 2$

25. $x = \cos t + t \sin t,\ y = \sin t - t \cos t,\ (t \ge 0)$. (See Example 9 of Section 8.2.)

8.4 Arc Lengths and Areas for Parametric Curves

In this section we look at the problems of finding lengths of curves defined parametrically, areas of surfaces of revolution obtained by rotating parametric curves, and areas of plane regions bounded by parametric curves.

Arc Lengths and Surface Areas

Let C be a smooth parametric curve with equations

$$x = f(t), \qquad y = g(t), \qquad (a \le t \le b).$$

(We assume that $f'(t)$ and $g'(t)$ are continuous on the interval $[a, b]$ and are never both zero.) From the differential triangle with legs dx and dy and hypotenuse ds (see Figure 8.27), we obtain $(ds)^2 = (dx)^2 + (dy)^2$, so we have

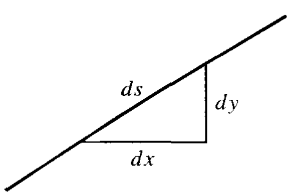


Figure 8.27 A differential triangle

The arc length element for a parametric curve

$$ds = \frac{ds}{dt}\,dt = \sqrt{\left(\frac{ds}{dt}\right)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The length of the curve C is given by

$$s = \int_{t=a}^{t=b} ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Example 1 Find the length of the parametric curve

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad (0 \le t \le 2).$$

Solution We have

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \qquad \frac{dy}{dt} = e^t(\sin t + \cos t).$$

Squaring these formulas, adding and simplifying, we get

$$\begin{aligned}\left(\frac{ds}{dt}\right)^2 &= e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 \\ &= e^{2t}\left(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t\right) \\ &= 2e^{2t}.\end{aligned}$$

The length of the curve is therefore

$$s = \int_0^2 \sqrt{2e^{2t}}\, dt = \sqrt{2}\int_0^2 e^t\, dt = \sqrt{2}\,(e^2 - 1) \text{ units.}$$

Parametric curves can be rotated around various axes to generate surfaces of revolution. The areas of these surfaces can be found by the same procedure used for graphs of functions, with the appropriate version of ds. If the curve

$$x = f(t), \qquad y = g(t), \qquad (a \le t \le b)$$

is rotated about the x-axis, the area S of the surface so generated is given by

$$S = 2\pi \int_{t=a}^{t=b} |y|\, ds = 2\pi \int_a^b |g(t)|\sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

If the rotation is about the y-axis, then the area is

$$S = 2\pi \int_{t=a}^{t=b} |x|\, ds = 2\pi \int_a^b |f(t)|\sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

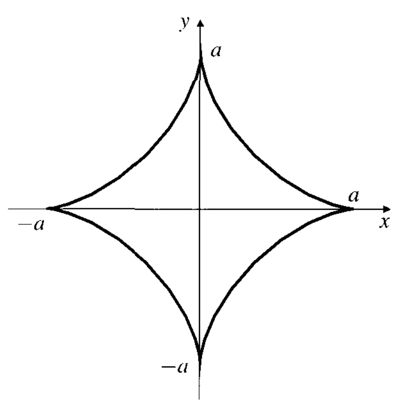


Figure 8.28

Example 2 Find the area of the surface of revolution obtained by rotating the astroid curve

$$x = a\cos^3 t, \qquad y = a \sin^3 t$$

(where $a > 0$) about the x-axis.

Solution The curve is symmetric about both coordinate axes. (See Figure 8.28.) The entire surface will be generated by rotating the upper half of the curve, and, in fact, we need only rotate the first quadrant part and multiply by 2. The first quadrant part of the curve corresponds to $0 \le t \le \pi/2$. We have

$$\frac{dx}{dt} = -3a\cos^2 t \sin t, \qquad \frac{dy}{dt} = 3a \sin^2 t \cos t.$$

Accordingly, the arc length element is

$$\begin{aligned}
ds &= \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt\\
&= 3a\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}\,dt\\
&= 3a\cos t\sin t\,dt.
\end{aligned}$$

Therefore, the required surface area is

$$\begin{aligned}
S &= 2\times 2\pi\int_0^{\pi/2} a\sin^3 t\,3a\cos t\sin t\,dt\\
&= 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt \qquad \text{Let } u=\sin t,\ du=\cos t\,dt.\\
&= 12\pi a^2\int_0^1 u^4\,du = \frac{12\pi a^2}{5}\ \text{square units.}
\end{aligned}$$

Areas Bounded by Parametric Curves

Consider the parametric curve C with equations $x = f(t)$, $y = g(t)$, $(a \le t \le b)$, where f is differentiable and g is continuous on $[a, b]$. For the moment, let us also assume that $f'(t) \ge 0$ and $g(t) \ge 0$ on $[a, b]$, so C has no points below the x-axis and is traversed from left to right as t increases from a to b.

The region under C and above the x-axis has area element given by $dA = y\,dx = g(t)f'(t)\,dt$, so its area (see Figure 8.29) is

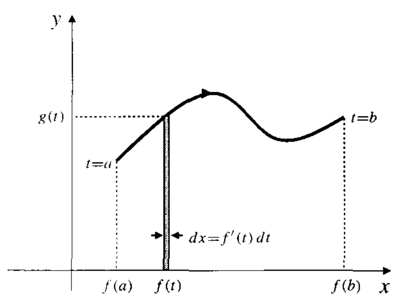


Figure 8.29

$$A = \int_a^b g(t)f'(t)\,dt.$$

Similar arguments can be given for three other cases:

If $f'(t) \ge 0$ and $g(t) \le 0$ on $[a, b]$, then $A = -\displaystyle\int_a^b g(t)f'(t)\,dt$,

If $f'(t) \le 0$ and $g(t) \ge 0$ on $[a, b]$, then $A = -\displaystyle\int_a^b g(t)f'(t)\,dt$,

If $f'(t) \le 0$ and $g(t) \le 0$ on $[a, b]$, then $A = \displaystyle\int_a^b g(t)f'(t)\,dt$,

where A is the (positive) area bounded by C, the x-axis, and the vertical lines $x = f(a)$ and $x = f(b)$. Combining these results we can see that

$$\int_a^b g(t)f'(t)\,dt = A_1 - A_2,$$

where A_1 is the area lying vertically between C and that part of the x-axis consisting of points $x = f(t)$ such that $g(t)f'(t) \ge 0$, and A_2 is a similar area corresponding to points where $g(t)f'(t) < 0$. This formula is valid for arbitrary continuous g and differentiable f. See Figure 8.30 for generic examples. In particular, if C is a non-self-intersecting closed curve, then the area of the region bounded by C is given

by

$$A = \int_a^b g(t)f'(t)\,dt \qquad \text{if } \mathcal{C} \text{ is traversed clockwise as } t \text{ increases,}$$

$$A = -\int_a^b g(t)f'(t)\,dt \qquad \text{if } \mathcal{C} \text{ is traversed counterclockwise,}$$

both of which are illustrated in Figure 8.31.

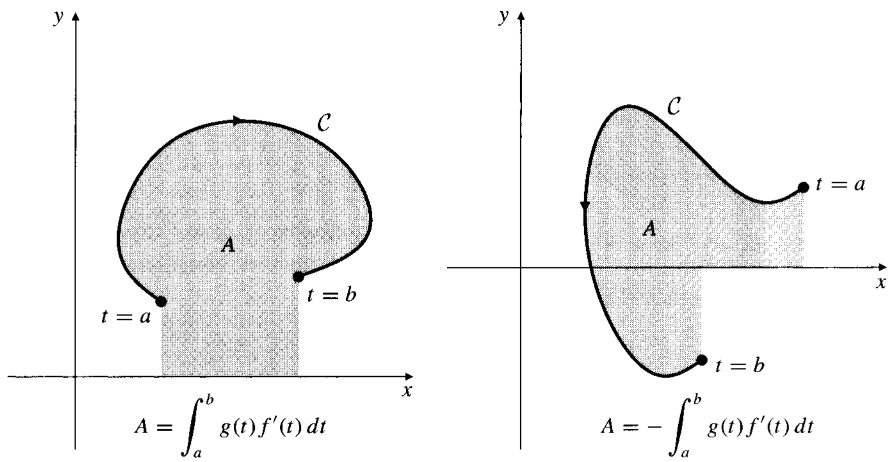


Figure 8.30 Areas defined by parametric curves

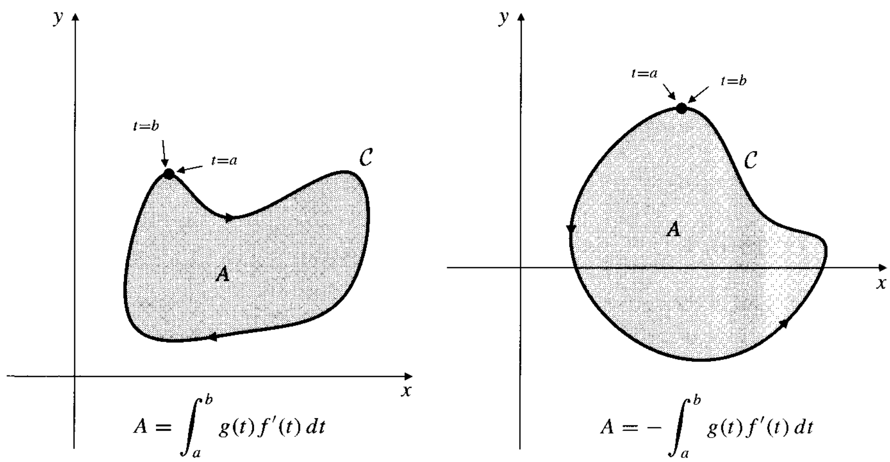


Figure 8.31 Areas bounded by closed parametric curves

Example 3 Find the area bounded by the ellipse $x = a\cos s$, $y = b\sin s$, $(0 \le s \le 2\pi)$.

Solution This ellipse is traversed counterclockwise. (See Example 4 in Section 8.2.) The area enclosed is

$$\begin{aligned} A &= -\int_0^{2\pi} b\sin s(-a\sin s)\,ds \\ &= \frac{ab}{2}\int_0^{2\pi}(1-\cos 2s)\,ds \\ &= \frac{ab}{2}s\bigg|_0^{2\pi} - \frac{ab}{4}\sin 2s\bigg|_0^{2\pi} = \pi ab \text{ square units.} \end{aligned}$$

■

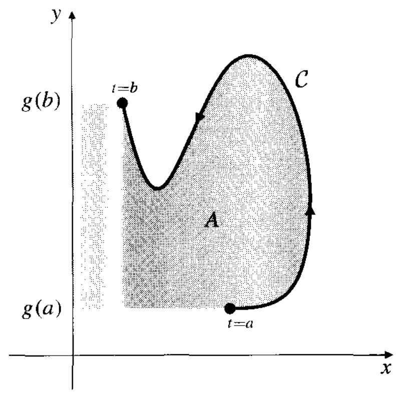


Figure 8.32 The shaded area is

$$A = \int_a^b f(t)g'(t)\,dt$$

Example 4 Find the area above the x-axis and under one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$.

Solution Part of the cycloid is shown in Figure 8.21 in Section 8.2. One arch corresponds to the parameter interval $0 \le t \le 2\pi$. Since $y = a(1-\cos t) \ge 0$ and $dx/dt = a(1-\cos t) \ge 0$, the area under one arch is

$$\begin{aligned} A &= \int_0^{2\pi} a^2(1-\cos t)^2\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt \\ &= a^2\left(t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_0^{2\pi} = 3\pi a^2 \text{ square units.} \end{aligned}$$

Similar arguments to those used above show that if f is continuous and g is differentiable, then we can also interpret

$$\int_a^b f(t)g'(t)\,dt = \int_{t=a}^{t=b} x\,dy = A_1 - A_2,$$

where A_1 is the area of the region lying *horizontally* between the parametric curve $x = f(t)$, $y = g(t)$, $(a \le t \le b)$ and that part of the y-axis consisting of points $y = g(t)$ such that $f(t)g'(t) \ge 0$, and A_2 is the area of a similar region corresponding to $f(t)g'(t) < 0$. For example, the region shaded in Figure 8.32 has area $\int_a^b f(t)g'(t)\,dt$. Green's Theorem in Section 16.3 provides a more coherent approach to finding such areas.

Exercises 8.4

Find the lengths of the curves in Exercises 1–8.

1. $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$
2. $x = 1 + t^3$, $y = 1 - t^2$, $(-1 \le t \le 2)$
3. $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$
4. $x = \ln(1+t^2)$, $y = 2\tan^{-1} t$, $(0 \le t \le 1)$
5. $x = t^2\sin t$, $y = t^2\cos t$, $(0 \le t \le 2\pi)$
6. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $(0 \le t \le 2\pi)$
7. $x = t + \sin t$, $y = \cos t$, $(0 \le t \le \pi)$
8. $x = \sin^2 t$, $y = 2\cos t$, $(0 \le t \le \pi/2)$
9. Find the length of one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (One arch corresponds to $0 \le t \le 2\pi$.)
10. Find the area of the surfaces obtained by rotating one arch of the cycloid in Exercise 9 about (a) the x-axis, (b) the y-axis.
11. Find the area of the surface generated by rotating the curve $x = e^t\cos t$, $y = e^t\sin t$, $(0 \le t \le \pi/2)$ about the x-axis.
12. Find the area of the surface generated by rotating the curve of Exercise 11 about the y-axis.
13. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the y-axis.
14. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the x-axis.

In Exercises 15–20, sketch and find the area of the region R described in terms of the given parametric curves.

15. R is the closed loop bounded by $x = t^3 - 4t$, $y = t^2$, $(-2 \le t \le 2)$.
16. R is bounded by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$.
17. R is bounded by the coordinate axes and the parabolic arc $x = \sin^4 t$, $y = \cos^4 t$.
18. R is bounded by $x = \cos s\sin s$, $y = \sin^2 s$, $(0 \le s \le \pi/2)$, and the y-axis.
19. R is bounded by the oval $x = (2 + \sin t)\cos t$, $y = (2 + \sin t)\sin t$.

* 20. R is bounded by the x-axis, the hyperbola $x = \sec t$, $y = \tan t$, and the ray joining the origin to the point $(\sec t_0, \tan t_0)$.

21. Show that the region bounded by the x-axis and the hyperbola $x = \cosh t$, $y = \sinh t$ (where $t > 0$), and the ray

from the origin to the point $(\cosh t_0, \sinh t_0)$ has area $t_0/2$ square units. This proves a claim made at the beginning of Section 3.6.

22. Find the volume of the solid obtained by rotating about the x-axis the region bounded by that axis and one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (See Example 8 in Section 8.2.)

23. Find the volume generated by rotating about the x-axis the region lying under the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ and above the x-axis.

8.5 Polar Coordinates and Polar Curves

The **polar coordinate system** is an alternative to the rectangular (Cartesian) coordinate system for describing the location of points in a plane. Sometimes it is more important to know how far, and in what direction, a point is from the origin than it is to know its Cartesian coordinates. In the polar coordinate system there is an origin (or **pole**), O, and a **polar axis**, a ray (i.e., a half-line) extending from O horizontally to the right. The position of any point P in the plane is then determined by its polar coordinates $[r, \theta]$, where

(i) r is the distance from O to P, and

(ii) θ is the angle that the ray OP makes with the polar axis (counterclockwise angles being considered positive).

We will use square brackets $[\cdot, \cdot]$ for polar coordinates of a point to distinguish them from rectangular (Cartesian) coordinates. Figure 8.33 shows some points with their polar coordinates. The rectangular coordinate axes x and y are usually shown on a polar graph. The polar axis coincides with the positive x-axis.

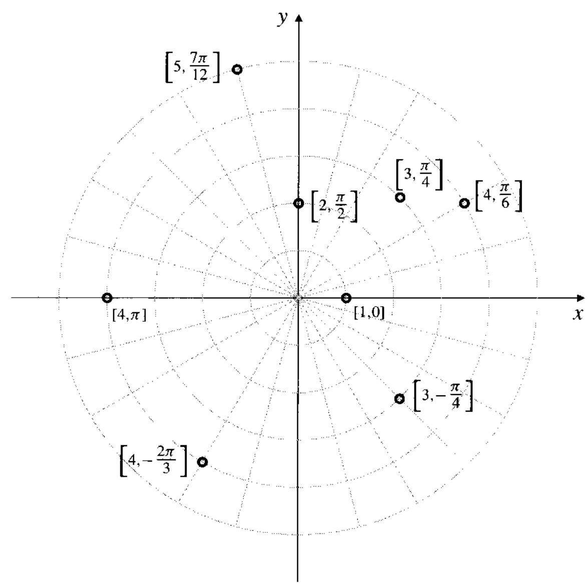


Figure 8.33 Polar coordinates of some points in the plane

Unlike rectangular coordinates, the polar coordinates of a point are not unique. The polar coordinates $[r, \theta_1]$ and $[r, \theta_2]$ represent the same point provided θ_1 and θ_2 differ by an integer multiple of 2π:

$$\theta_2 = \theta_1 + 2n\pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \ldots.$$

For instance, the polar coordinates

$$\left[3, \frac{\pi}{4}\right], \quad \left[3, \frac{9\pi}{4}\right], \quad \text{and} \quad \left[3, -\frac{7\pi}{4}\right]$$

all represent the same point with Cartesian coordinates $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$. Similarly, $[4, \pi]$ and $[4, -\pi]$ both represent the point with Cartesian coordinates $(-4, 0)$, and $[1, 0]$ and $[1, 2\pi]$ both represent the point with Cartesian coordinates $(1, 0)$. In addition, the origin O has polar coordinates $[0, \theta]$ for any value of θ. (If we go zero distance from O, it doesn't matter in what direction we go.)

Sometimes we need to interpret polar coordinates $[r, \theta]$, where $r < 0$. The appropriate interpretation for this "negative distance" r is that it represents a positive distance $-r$ measured in the *opposite direction* (i.e., in the direction $\theta + \pi$):

$$[r, \theta] = [-r, \theta + \pi].$$

For example, $[-1, \pi/4] = [1, 5\pi/4]$. Allowing $r < 0$ increases the number of different sets of polar coordinates that represent the same point.

If we want to consider both rectangular and polar coordinate systems in the same plane, and we choose the positive x-axis as the polar axis, then the relationships between the rectangular coordinates of a point and its polar coordinates are as shown in Figure 8.34.

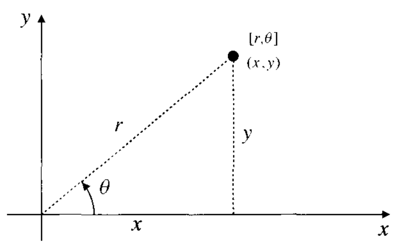


Figure 8.34 Relating Cartesian and polar coordinates of a point

Polar–rectangular conversion

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

A single equation in x and y generally represents a curve in the plane with respect to the Cartesian coordinate system. Similarly, a single equation in r and θ generally represents a curve with respect to the polar coordinate system. The conversion formulas above can be used to convert one representation of a curve into the other.

Example 1 The straight line $2x - 3y = 5$ has polar equation $r(2\cos\theta - 3\sin\theta) = 5$, or

$$r = \frac{5}{2\cos\theta - 3\sin\theta}.$$

Example 2 Find the Cartesian equation of the curve represented by the polar equation $r = 2a\cos\theta$; hence identify the curve.

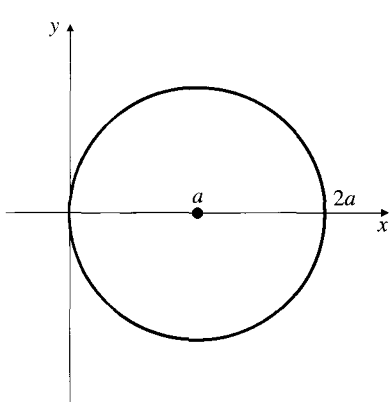


Figure 8.35 The circle $r = 2a\cos\theta$

Solution The polar equation can be transformed to Cartesian coordinates if we first multiply it by r:

$$r^2 = 2ar\cos\theta$$
$$x^2 + y^2 = 2ax$$
$$(x - a)^2 + y^2 = a^2$$

The given polar equation $r = 2a\cos\theta$ thus represents a circle with centre $(a, 0)$ and radius a as shown in Figure 8.35. Observe from the equation that $r \to 0$ as $\theta \to \pm\pi/2$. In the figure, this corresponds to the fact that the circle approaches the origin in the vertical direction.

Some Polar Curves

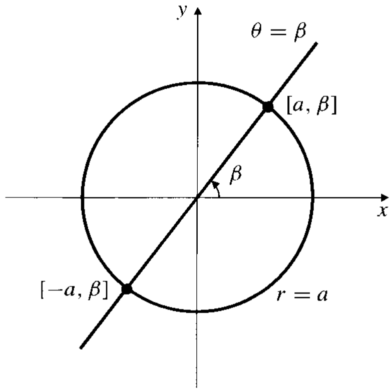


Figure 8.36 Coordinate curves for the polar coordinate system

Figure 8.36 shows the graphs of the polar equations $r = a$ and $\theta = \beta$, where a and β (Greek *beta*) are constants. These are, respectively, the circle with radius $|a|$ centred at the origin, and a line through the origin making angle β with the polar axis. Note that the line and the circle meet at two points, with polar coordinates $[a, \beta]$ and $[-a, \beta]$. The "coordinate curves" for polar coordinates, that is, the curves with equations $r =$ constant and $\theta =$ constant, are circles centred at the origin and lines through the origin, respectively. The "coordinate curves" for Cartesian coordinates, $x =$ constant and $y =$ constant, are vertical and horizontal straight lines. Cartesian graph paper is ruled with vertical and horizontal lines; polar graph paper is ruled with concentric circles and radial lines emanating from the origin, as shown in Figure 8.33.

The graph of an equation of the form $r = f(\theta)$ is called the **polar graph** of the function f. Some polar graphs can be recognized easily if the polar equation is transformed to rectangular form. For others, this transformation does not help; the rectangular equation may be too complicated to be recognizable. In these cases one must resort to constructing a table of values and plotting points.

Example 3 Sketch and identify the curve $r = 2a\cos(\theta - \theta_0)$.

Solution We proceed as in Example 2.

$$\begin{aligned}
r^2 &= 2ar\cos(\theta - \theta_0) = 2ar\cos\theta_0\cos\theta + 2ar\sin\theta_0\sin\theta \\
x^2 + y^2 &= 2a\cos\theta_0 x + 2a\sin\theta_0 y \\
x^2 - 2a\cos\theta_0 x &+ a^2\cos^2\theta_0 + y^2 - 2a\sin\theta_0 y + a^2\sin^2\theta_0 = a^2 \\
(x - a\cos\theta_0)^2 &+ (y - a\sin\theta_0)^2 = a^2.
\end{aligned}$$

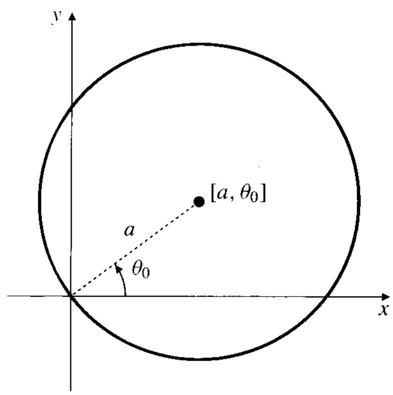


Figure 8.37 The circle $r = 2a\cos(\theta - \theta_0)$

This is a circle of radius a that passes through the origin in the directions $\theta = \theta_0 \pm \frac{\pi}{2}$, which make $r = 0$. (See Figure 8.37.) Its centre has Cartesian coordinates $(a\cos\theta_0, a\sin\theta_0)$ and hence polar coordinates $[a, \theta_0]$. For $\theta_0 = \pi/2$ we have $r = 2a\sin\theta$ as the equation of a circle of radius a centred on the y-axis.

Comparing Examples 2 and 3, we are led to formulate the following principle.

Rotating a polar graph

The polar graph with equation $r = f(\theta - \theta_0)$ is the polar graph with equation $r = f(\theta)$ rotated through angle θ_0 about the origin.

Example 4 Sketch the polar curve $r = a(1 - \cos\theta)$, where $a > 0$.

Solution Transformation to rectangular coordinates is not much help here; the resulting equation is $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$ (verify this), which we do not recognize. Therefore, we will make a table of values and plot some points.

Table 3.

θ	0	$\pm\frac{\pi}{6}$	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{3}$	$\pm\frac{\pi}{2}$	$\pm\frac{2\pi}{3}$	$\pm\frac{3\pi}{4}$	$\pm\frac{5\pi}{6}$	π
r	0	$0.13a$	$0.29a$	$0.5a$	a	$1.5a$	$1.71a$	$1.87a$	$2a$

Because it is shaped like a heart, this curve is called a **cardioid**. Observe the cusp at the origin in Figure 8.38. As in the previous example, the curve enters the origin in the directions θ that make $r = f(\theta) = 0$. In this case, the only such direction is $\theta = 0$. It is important, when sketching polar graphs, to show clearly any directions of approach to the origin.

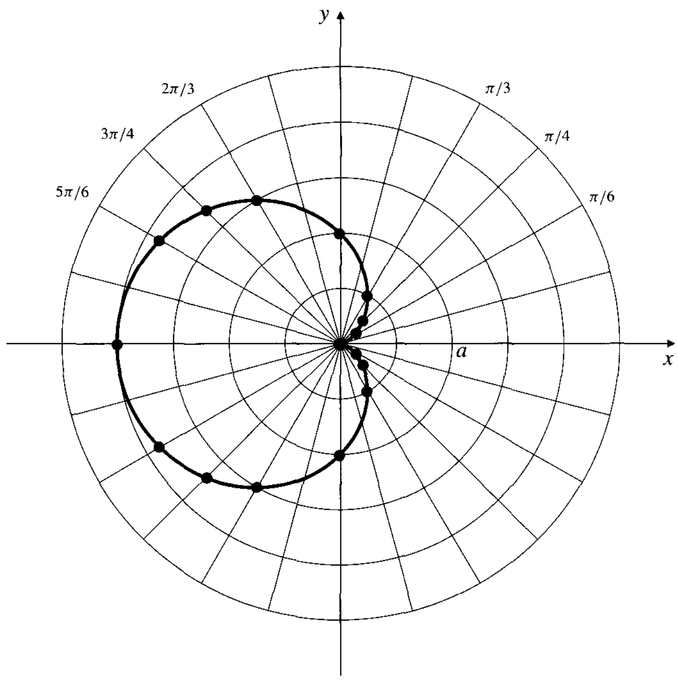


Figure 8.38 The cardioid $r = a(1 - \cos\theta)$

Direction of a polar graph at the origin

A polar graph $r = f(\theta)$ approaches the origin from the direction θ for which $f(\theta) = 0$.

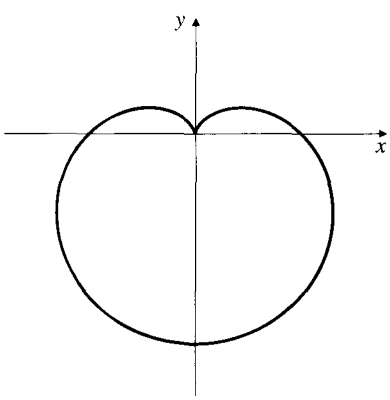


Figure 8.39 The cardioid $r = a(1 - \sin\theta)$

The equation $r = a(1 - \cos(\theta - \theta_0))$ represents a cardioid of the same size and shape as that in Figure 8.38 but rotated through an angle θ_0 counterclockwise about the origin. Its cusp is in the direction $\theta = \theta_0$. In particular, $r = a(1 - \sin\theta)$ has a vertical cusp, as shown in Figure 8.39.

It is not usually necessary to make a detailed table of values to sketch a polar curve with a simple equation of the form $r = f(\theta)$. It is essential to determine those values of θ for which $r = 0$ and indicate them on the graph with rays. It is also useful to determine points where the curve is farthest from the origin. (Where is $f(\theta)$ maximum or minimum?) Except possibly at the origin, polar curves will be smooth wherever $f(\theta)$ is a differentiable function of θ.

Example 5 Sketch the polar graphs (a) $r = \cos(2\theta)$, (b) $r = \sin(3\theta)$, and (c) $r^2 = \cos(2\theta)$.

Solution The graphs are shown in Figures 8.40–8.42. Observe how the curves (a) and (c) approach the origin in the directions $\theta = \pm\frac{\pi}{4}$ and $\theta = \pm\frac{3\pi}{4}$, and curve

(b) approaches in the directions $\theta = 0,\ \pi,\ \pm\frac{\pi}{3}$ and $\pm\frac{2\pi}{3}$. This curve is traced out twice as θ increases from $-\pi$ to π. So is curve (c) if we allow both square roots $r = \pm\sqrt{\cos(2\theta)}$. Note that there are no points on curve (c) between $\theta = \pm\frac{\pi}{4}$ and $\theta = \pm\frac{3\pi}{4}$ because r^2 cannot be negative.

Curve (c) is called a **lemniscate**. Lemniscates are curves consisting of points P such that the product of the distances from P to certain fixed points is constant. For the curve (c) these fixed points are $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$.

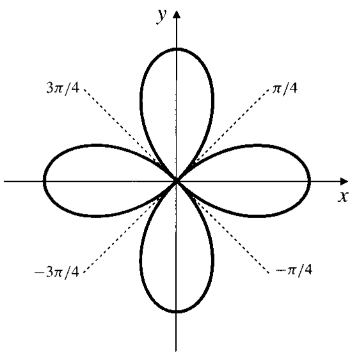


Figure 8.40 Curve (a): the polar curve $r = \cos(2\theta)$

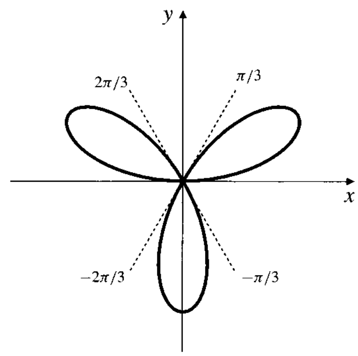


Figure 8.41 Curve (b): the polar curve $r = \sin(3\theta)$

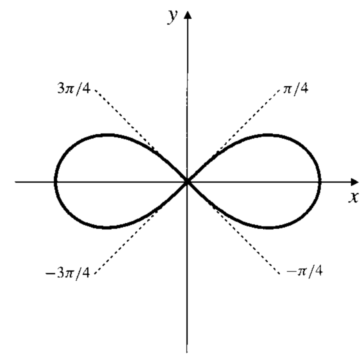
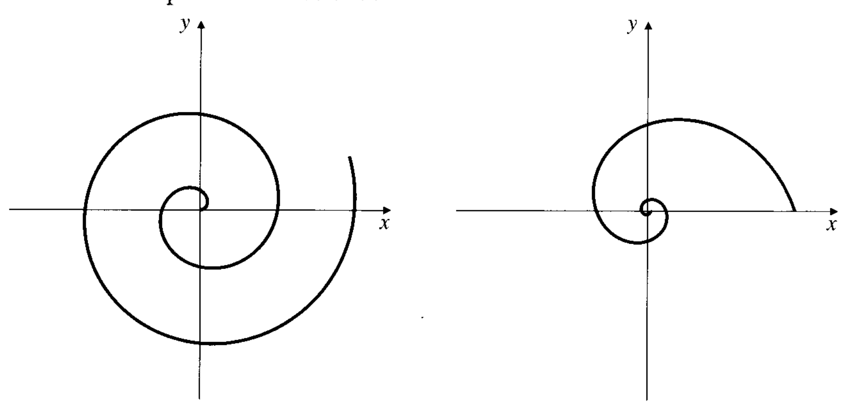


Figure 8.42 Curve (c): the lemniscate $r^2 = \cos(2\theta)$

In all of the examples above, the functions $f(\theta)$ are periodic and 2π is a period of each of them, so each line through the origin could meet the polar graph at most twice. (θ and $\theta+\pi$ determine the same line.) If $f(\theta)$ does not have period 2π, then the curve can wind around the origin many times. Two such *spirals* are shown in Figure 8.43, the **equiangular spiral** $r = \theta$ and the **exponential spiral** $r = e^{-\theta/3}$, each sketched for positive values of θ.



(a) (b)

Figure 8.43
(a) The equiangular spiral $r = \theta$
(b) The exponential spiral $r = e^{-\theta/3}$

Remark Maple has a `polarplot` routine as part of its "plots" package, which must be loaded prior to the use of polarplot. Here is how to get Maple to plot on the same graph the polar curves $r = 1$ and $r = 2\sin(3\theta)$, for $0 \le \theta \le 2\pi$:

```
> with(plots):
> polarplot([1,2*sin(3*t)],t=0..2*Pi,scaling=constrained);
```

The option `scaling=constrained` is necessary with polar plots to force Maple to use the same distance unit on both axes (so a circle will appear circular).

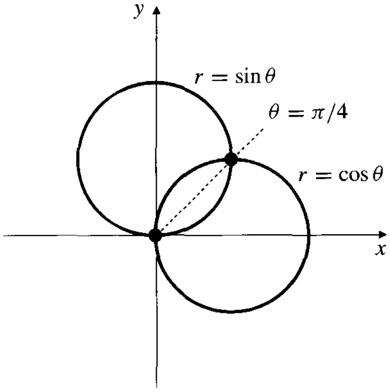


Figure 8.44

Intersections of Polar Curves

Because the polar coordinates of points are not unique, finding the intersection points of two polar curves can be more complicated than the similar problem for Cartesian graphs. Of course, the polar curves $r = f(\theta)$ and $r = g(\theta)$ will intersect at any points $[r_0, \theta_0]$ for which

$$f(\theta_0) = g(\theta_0) \qquad \text{and} \qquad r_0 = f(\theta_0),$$

but there may be other intersections as well. In particular, if both curves pass through the origin, then the origin will be an intersection point, even though it may not show up in solving $f(\theta) = g(\theta)$, because the curves may be at the origin for different values of θ. For example, the two circles $r = \cos\theta$ and $r = \sin\theta$ intersect at the origin and also at the point $[1/\sqrt{2}, \pi/4]$, even though only the latter point is obtained by solving the equation $\cos\theta = \sin\theta$. (See Figure 8.44.)

Example 6 Find the intersections of the curves $r = \sin\theta$ and $r = 1 - \sin\theta$.

Solution Since both functions of θ are periodic with period 2π, we need only look for solutions satisfying $0 \le \theta \le 2\pi$. Solving the equation

$$\sin\theta = 1 - \sin\theta,$$

we get $\sin\theta = 1/2$, so that $\theta = \pi/6$ or $\theta = 5\pi/6$. Both curves have $r = 1/2$ at these points, so the two curves intersect at $[1/2, \pi/6]$ and $[1/2, 5\pi/6]$. Also, the origin lies on the curve $r = \sin\theta$ (for $\theta = 0$ and $\theta = 2\pi$) and on the curve $r = 1 - \sin\theta$ (for $\theta = \pi/2$). Therefore, the origin is also an intersection point of the curves. (See Figure 8.45.)

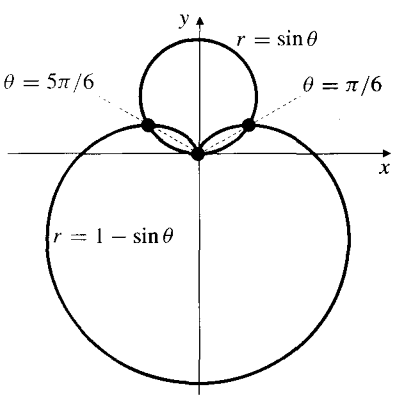


Figure 8.45

Finally, if negative values of r are allowed, then the curves $r = f(\theta)$, $y = g(\theta)$ will also intersect at $[r_1, \theta_1] = [r_2, \theta_2]$ if, for some integer k,

$$\theta_1 = \theta_2 + (2k+1)\pi \qquad \text{and} \qquad r_1 = f(\theta_1) = -g(\theta_2) = -r_2.$$

See Exercise 28 for an example.

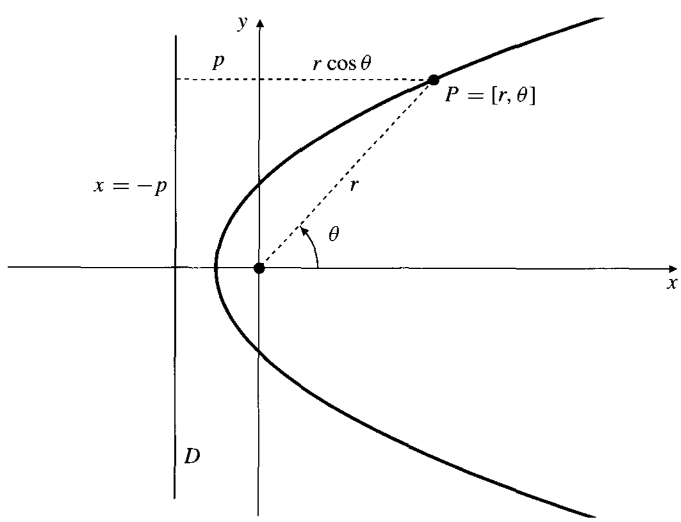


Figure 8.46 A conic curve with eccentricity ε, focus at the origin, and directrix $x = -p$

Polar Conics

Let D be the vertical straight line $x = -p$ and let ε be a positive real number. The set of points P in the plane that satisfy the condition

$$\frac{\text{distance of } P \text{ from the origin}}{\text{perpendicular distance from } P \text{ to } D} = \varepsilon$$

is a conic section with eccentricity ε, focus at the origin, and corresponding directrix D, as observed in Section 8.1. (It is an ellipse if $\varepsilon < 1$, a parabola if $\varepsilon = 1$, and a hyperbola if $\varepsilon > 1$.) If P has polar coordinates $[r, \theta]$, then the condition above becomes (see Figure 8.46)

$$\frac{r}{p + r\cos\theta} = \varepsilon,$$

or, solving for r,

$$r = \frac{\varepsilon p}{1 - \varepsilon\cos\theta}.$$

Examples of the three possibilities (ellipse, parabola, and hyperbola) are shown in Figures 8.47–8.49. Note that for the hyperbola, the directions of the asymptotes are the angles that make the denominator $1 - \varepsilon\cos\theta = 0$. We will have more to say about polar equations of conics, especially ellipses, in Section 11.6.

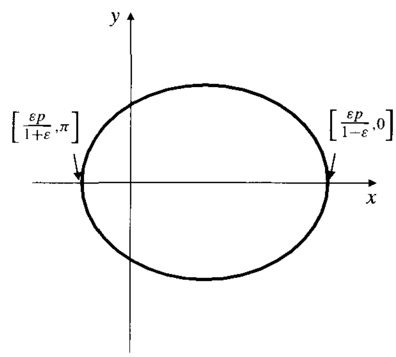


Figure 8.47 Ellipse: $\varepsilon < 1$

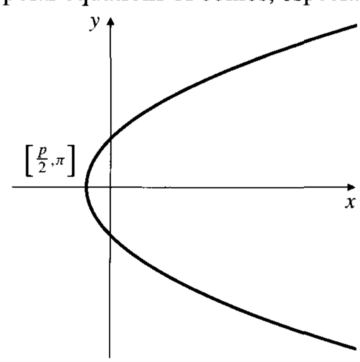


Figure 8.48 Parabola: $\varepsilon = 1$

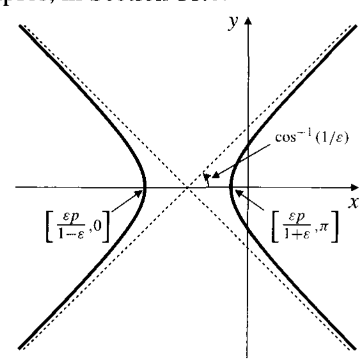


Figure 8.49 Hyperbola: $\varepsilon > 1$

Exercises 8.5

In Exercises 1–12, transform the given polar equation to rectangular coordinates and thus identify the curve represented.

1. $r = 3\sec\theta$

2. $r = -2\csc\theta$

3. $r = \dfrac{5}{3\sin\theta - 4\cos\theta}$

4. $r = \sin\theta + \cos\theta$

5. $r^2 = \csc 2\theta$

6. $r = \sec\theta\tan\theta$

7. $r = \sec\theta(1 + \tan\theta)$

8. $r = \dfrac{2}{\sqrt{\cos^2\theta + 4\sin^2\theta}}$

9. $r = \dfrac{1}{1 - \cos\theta}$

10. $r = \dfrac{2}{2 - \cos\theta}$

11. $r = \dfrac{2}{1 - 2\sin\theta}$

12. $r = \dfrac{2}{1 + \sin\theta}$

In Exercises 13–24, sketch the polar graphs of the given equations.

13. $r = 1 + \sin\theta$

14. $r = 1 - \cos(\theta + \frac{\pi}{4})$

15. $r = 1 + 2\cos\theta$

16. $r = 1 - 2\sin\theta$

17. $r = 2 + \cos\theta$

18. $r = 2\sin 2\theta$

19. $r = \cos 3\theta$

20. $r = 2\cos 4\theta$

21. $r^2 = 4\sin 2\theta$

22. $r^2 = 4\cos 3\theta$

23. $r^2 = \sin 3\theta$

24. $r = \ln\theta$

Find all intersections of the pairs of curves in Exercises 25–28.

25. $r = \sqrt{3}\cos\theta, \quad r = \sin\theta$

26. $r^2 = 2\cos(2\theta), \quad r = 1$

27. $r = 1 + \cos\theta, \quad r = 3\cos\theta$

* **28.** $r = \theta, \quad r = \theta + \pi$

* **29.** Sketch the graph of the equation $r = 1/\theta$, $\theta > 0$. Show that this curve has a horizontal asymptote. Does $r = 1/(\theta - \alpha)$ have an asymptote?

30. How many leaves does the curve $r = \cos n\theta$ have? the curve $r^2 = \cos n\theta$? Distinguish the cases where n is odd and even.

31. Show that the polar graph $r = f(\theta)$ (where f is continuous) can be written as a parametric curve with parameter θ.

In Exercises 32–37, use computer graphing software or a graphing calculator to plot various members of the given families of polar curves, and try to observe patterns that would enable you to predict behaviour of other members of the families.

32. $r = \cos\theta\cos(m\theta), \quad m = 1, 2, 3, \ldots$

33. $r = 1 + \cos\theta\cos(m\theta), \quad m = 1, 2, 3, \ldots$

34. $r = \sin(2\theta)\sin(m\theta), \quad m = 2, 3, 4, 5, \ldots$

35. $r = 1 + \sin(2\theta)\sin(m\theta), \quad m = 2, 3, 4, 5, \ldots$

36. $r = C + \cos\theta\cos(2\theta)$ for $C = 0$, $C = 1$, values of C between 0 and 1, and values of C greater than 1

37. $r = C + \cos\theta\sin(3\theta)$ for $C = 0$, $C = 1$, values of C between 0 and 1, values of C less than 0, and values of C greater than 1

38. Plot the curve $r = \ln\theta$ for $0 < \theta \le 2\pi$. It intersects itself at point P. Thus there are two values θ_1 and θ_2 between 0 and 2π for which $[f(\theta_1), \theta_1] = [f(\theta_2), \theta_2]$. What equations must be satsified by θ_1 and θ_2? Find θ_1 and θ_2, and find the Cartesian coordinates of P correct to 6 decimal places.

39. Simultaneously plot the two curves $r = \ln\theta$ and $r = 1/\theta$, for $0 < \theta \le 2\pi$. The two curves intersect at two points. What equations must be satisfied by the θ values of these points? What are their Cartesian coordinates to 6 decimal places?

8.6 Slopes, Areas, and Arc Lengths for Polar Curves

There is a simple formula that can be used to determine the direction of the tangent line to a polar curve $r = f(\theta)$ at a point $P = [r, \theta]$ other than the origin. Let Q be a point on the curve near P corresponding to polar angle $\theta + h$. Draw PS perpendicular to OQ. Observe that $PS = f(\theta)\sin h$ and $SQ = OQ - OS = f(\theta + h) - f(\theta)\cos h$. If the tangent line to $r = f(\theta)$ at P makes angle ψ (Greek *psi*) with the radial line OP as shown in Figure 8.50(a), then ψ is the limit of the angle SQP as $h \to 0$. Thus

$$\begin{aligned} \tan\psi = \lim_{h\to 0}\frac{PS}{SQ} &= \lim_{h\to 0}\frac{f(\theta)\sin h}{f(\theta+h) - f(\theta)\cos h} && \left[\frac{0}{0}\right] \\ &= \lim_{h\to 0}\frac{f(\theta)\cos h}{f'(\theta+h) + f(\theta)\sin h} && \text{(by l'Hôpital's Rule)} \\ &= \frac{f(\theta)}{f'(\theta)} = \frac{r}{dr/d\theta}. \end{aligned}$$

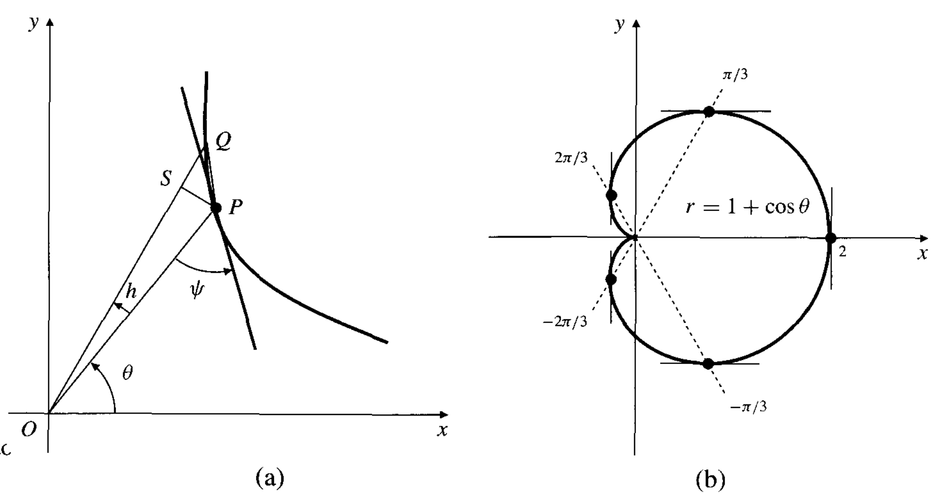


Figure 8.50

(a) The angle ψ is the limit of angle SQP as $h \to 0$

(b) Horizontal and vertical tangents to a cardioid

Tangent direction for a polar curve

At any point P other than the origin on the polar curve $r = f(\theta)$, the angle ψ between the radial line from the origin to P and the tangent to the curve is given by

$$\tan\psi = \frac{f(\theta)}{f'(\theta)}.$$

In particular, $\psi = \pi/2$ if $f'(\theta) = 0$.
If $f(\theta_0) = 0$ and the curve has a tangent line at θ_0, then that tangent line has equation $\theta = \theta_0$.

The formula above can be used to find points where a polar graph has horizontal or vertical tangents:

$$\psi + \theta = \pi, \quad \text{so } \tan\psi = -\tan\theta \quad \text{for a horizontal tangent,}$$
$$\psi + \theta = \frac{\pi}{2}, \quad \text{so } \tan\psi = \cot\theta \quad \text{for a vertical tangent.}$$

Remark Since for parametric curves horizontal and vertical tangents correspond to $dy/dt = 0$ and $dx/dt = 0$, respectively, it is usually easier to find the critical points of $y = f(\theta)\sin\theta$ for horizontal tangents and of $x = f(\theta)\cos\theta$ for vertical tangents.

Example 1 Find the points on the cardioid $r = 1 + \cos\theta$, where the tangent lines are vertical or horizontal.

Solution We have $y = (1+\cos\theta)\sin\theta$ and $x = (1+\cos\theta)\cos\theta$. For horizontal tangents

$$\begin{aligned} 0 = \frac{dy}{d\theta} &= -\sin^2\theta + \cos^2\theta + \cos\theta \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= (2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The solutions are $\cos\theta = \frac{1}{2}$ and $\cos\theta = -1$, that is, $\theta = \pm\pi/3$ and $\theta = \pi$. There are horizontal tangents at $\left[\frac{3}{2}, \pm\frac{\pi}{3}\right]$. At $\theta = \pi$, we have $r = 0$. The curve does not have a tangent line at the origin (it has a cusp). See Figure 8.50(b).

For vertical tangents

$$0 = \frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta(1 + 2\cos\theta).$$

The solutions are $\sin\theta = 0$ and $\cos\theta = -\frac{1}{2}$, that is, $\theta = 0, \pi, \pm 2\pi/3$. There are vertical tangent lines at $[2, 0]$ and $\left[\frac{1}{2}, \pm\frac{2\pi}{3}\right]$. ∎

Areas Bounded by Polar Curves

The basic area problem in polar coordinates is that of finding the area A of the region R bounded by the polar graph $r = f(\theta)$ and the two rays $\theta = \alpha$ and $\theta = \beta$. We assume that $\beta > \alpha$ and that f is continuous for $\alpha \le \theta \le \beta$. See Figure 8.51.

Figure 8.51 An area element in polar coordinates

A suitable area element in this case is a sector of angular width $d\theta$, as shown in Figure 8.51. For infinitesimal $d\theta$ this is just a sector of a circle of radius $r = f(\theta)$:

$$dA = \frac{d\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\,d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

$$A = \frac{1}{2}\int_\alpha (f(\theta))^2\,d\theta.$$

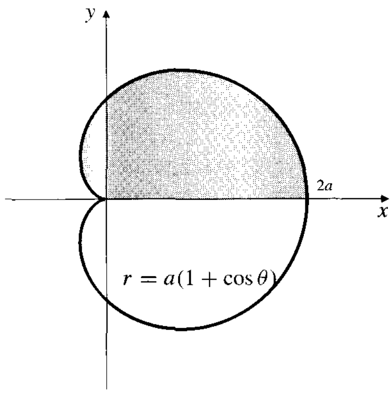


Figure 8.52

Example 2 Find the area bounded by the cardioid $r = a(1+\cos\theta)$, as illustrated in Figure 8.52.

Solution By symmetry, the area is twice that of the top half:

$$\begin{aligned} A &= 2\times\frac{1}{2}\int_0^\pi a^2(1+\cos\theta)^2\,d\theta \\ &= a^2\int_0^\pi (1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= a^2\int_0^\pi \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= a^2\left(\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right)\Bigg|_0^\pi = \frac{3}{2}\pi a^2 \text{ square units.} \end{aligned}$$

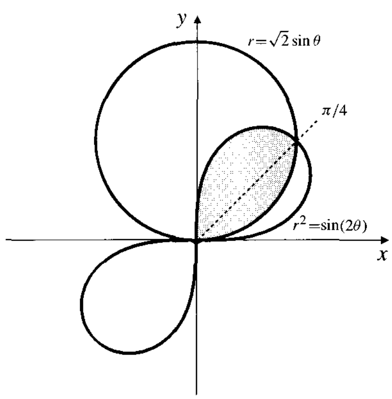


Figure 8.53

Example 3 Find the area of the region that lies inside the circle $r = \sqrt{2}\sin\theta$ and inside the lemniscate $r^2 = \sin 2\theta$.

Solution The region is shaded in Figure 8.53. Besides intersecting at the origin, the curves intersect at the first quadrant point satisfying

$$2\sin^2\theta = \sin 2\theta = 2\sin\theta\cos\theta.$$

Thus $\sin\theta = \cos\theta$ and $\theta = \pi/4$. The required area is

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/4} 2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\sin 2\theta\,d\theta \\ &= \int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta - \frac{1}{4}\cos 2\theta\Bigg|_{\pi/4}^{\pi/2} \\ &= \frac{\pi}{8} - \frac{1}{4}\sin 2\theta\Bigg|_0^{\pi/4} + \frac{1}{4} = \frac{\pi}{8} - \frac{1}{4} + \frac{1}{4} = \frac{\pi}{8} \quad \text{square units.} \end{aligned}$$

Arc Lengths for Polar Curves

The arc length element for the polar curve $r = f(\theta)$ can be determined from the

differential triangle shown in Figure 8.54. The leg $r\,d\theta$ of the triangle is obtained as the arc length of a circular arc of radius r subtending angle $d\theta$ at the origin. We have

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 = \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right](d\theta)^2,$$

so we obtain the following formula:

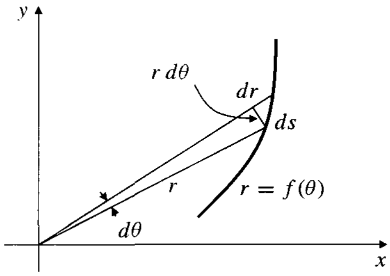


Figure 8.54 The arc length element for a polar curve

Arc length element for a polar curve

The arc length element for the polar curve $r = f(\theta)$ is

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \sqrt{\big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2}\,d\theta.$$

This arc length element can also be derived from that for a parametric curve. See Exercise 26 at the end of this section.

Example 4 Find the total length of the cardioid $r = a(1 + \cos\theta)$.

Solution The total length is twice the length from $\theta = 0$ to $\theta = \pi$. (Review Figure 8.52.) Since $dr/d\theta = -a\sin\theta$ for the cardioid, the arc length is

$$\begin{aligned} s &= 2\int_0^\pi \sqrt{a^2\sin^2\theta + a^2(1+\cos\theta)^2}\,d\theta \\ &= 2\int_0^\pi \sqrt{2a^2 + 2a^2\cos\theta}\,d\theta \qquad (\text{but } 1+\cos\theta = 2\cos^2(\theta/2)) \\ &= 2\sqrt{2}a\int_0^\pi \sqrt{2\cos^2\frac{\theta}{2}}\,d\theta \\ &= 4a\int_0^\pi \cos\frac{\theta}{2}\,d\theta = 8a\sin\frac{\theta}{2}\bigg|_0^\pi = 8a \text{ units.} \end{aligned}$$

Exercises 8.6

In Exercises 1–11, sketch and find the areas of the given polar regions R.

1. R lies between the origin and the spiral $r = \sqrt{\theta}$, $0 \le \theta \le 2\pi$.
2. R lies between the origin and the spiral $r = \theta$, $0 \le \theta \le 2\pi$.
3. R is bounded by the curve $r^2 = a^2\cos 2\theta$.
4. R is one leaf of the curve $r = \sin 3\theta$.
5. R is bounded by the curve $r = \cos 4\theta$.
6. R lies inside both of the circles $r = a$ and $r = 2a\cos\theta$.
7. R lies inside the cardioid $r = 1 - \cos\theta$ and outside the circle $r = 1$.
8. R lies inside the cardioid $r = a(1 - \sin\theta)$ and inside the circle $r = a$.
9. R lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.
10. R is bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$.
11. R is bounded by the smaller loop of the curve $r = 1 + 2\cos\theta$.

Find the lengths of the polar curves in Exercises 12–14

12. $r = \theta^2$, $0 \le \theta \le \pi$
13. $r = e^{a\theta}$, $-\pi \le \theta \le \pi$
14. $r = a\theta$, $0 \le \theta \le 2\pi$

15. Show that the total arc length of the lemniscate $r^2 = \cos 2\theta$ is $4\int_0^{\pi/4} \sqrt{\sec 2\theta}\, d\theta$.

16. One leaf of the lemniscate $r^2 = \cos 2\theta$ is rotated (a) about the x-axis and (b) about the y-axis. Find the area of the surface generated in each case.

* **17.** Determine the angles at which the straight line $\theta = \pi/4$ intersects the cardioid $r = 1 + \sin\theta$.

* **18.** At what points do the curves $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$ intersect? At what angle do the curves intersect at each of these points?

* **19.** At what points do the curves $r = 1 - \cos\theta$ and $r = 1 - \sin\theta$ intersect? At what angle do the curves intersect at each of these points?

In Exercises 20–25, find all points on the given curve where the tangent line is horizontal, vertical, or does not exist.

* **20.** $r = \cos\theta + \sin\theta$ * **21.** $r = 2\cos\theta$

* **22.** $r^2 = \cos 2\theta$ * **23.** $r = \sin 2\theta$

* **24.** $r = e^\theta$ * **25.** $r = 2(1 - \sin\theta)$

26. The polar curve $r = f(\theta)$, $(\alpha \le \theta \le \beta)$ can be parametrized:

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Derive the formula for the arc length element for the polar curve from that for a parametric curve.

Chapter Review

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ a conic section ◇ an ellipse
  - ◇ a parabola ◇ a hyperbola
  - ◇ a parametric curve ◇ a parametrization of a curve
  - ◇ a smooth curve ◇ a polar curve
- **What is the focus-directrix definition of a conic?**
- **How can you find the slope of a parametric curve?**
- **How can you find the length of a parametric curve?**
- **How can you find the length of a polar curve?**
- **How can you find the area bounded by a polar curve?**

Review Exercises

In Exercises 1–4, describe the conic having the given equation. Give its foci and principal axes and, if it is a hyperbola, its asymptotes.

1. $x^2 + 2y^2 = 2$ **2.** $9x^2 - 4y^2 = 36$

3. $x + y^2 = 2y + 3$ **4.** $2x^2 + 8y^2 = 4x - 48y$

Identify the parametric curves in Exercises 5–10.

5. $x = t,\ y = 2 - t,\ (0 \le t \le 2)$

6. $x = 2\sin 3t,\ y = 2\cos 3t,\ (0 \le t \le 1/2)$

7. $x = \cosh t,\ y = \sinh^2 t,$

8. $x = e^t,\ y = e^{-2t},\ (-1 \le t \le 1)$

9. $x = \cos(t/2),\ y = 4\sin(t/2),\ (0 \le t \le \pi)$

10. $x = \cos t + \sin t,\ y = \cos t - \sin t,\ (0 \le t \le 2\pi)$

In Exercises 11–14, determine the points where the given parametric curves have horizontal and vertical tangents, and sketch the curves.

11. $x = \dfrac{4}{1+t^2},\ y = t^3 - 3t$

12. $x = t^3 - 3t,\ y = t^3 + 3t$

13. $x = t^3 - 3t,\ y = t^3$

14. $x = t^3 - 3t,\ y = t^3 - 12t$

15. Find the area bounded by the part of the curve $x = t^3 - t$, $y = |t^3|$ that forms a closed loop.

16. Find the volume of the solid of revolution generated by rotating the closed loop in the previous exercise about the y-axis.

17. Find the length of the curve $x = e^t - t$, $y = 4e^{t/2}$ from $t = 0$ to $t = 2$.

18. Find the area of the surface obtained by rotating the arc in the previous exercise about the x-axis.

Sketch the polar graphs of the equations in Exercises 19–24.

19. $r = \theta,\ \left(-\frac{3\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$ **20.** $r = |\theta|,\ (-2\pi \le \theta \le 2\pi)$

21. $r = 1 + \cos 2\theta$ **22.** $r = 2 + \cos 2\theta$

23. $r = 1 + 2\cos 2\theta$ **24.** $r = 1 - \sin 3\theta$

25. Find the area of one of the two larger loops of the curve in Exercise 23.

26. Find the area of one of the two smaller loops of the curve in Exercise 23.

27. Find the area of the smaller of the two loops enclosed by the curve $r = 1 + \sqrt{2}\sin\theta$.

28. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and to the left of the line $x = 1/4$.

Challenging Problems

1. A glass in the shape of a circular cylinder of radius 4 cm is more than half filled with water. If the glass is tilted by an angle θ from the vertical, where θ is small enough that no water spills out, find the surface area of the water.

2. Show that a plane that is not parallel to the axis of a circular cylinder intersects the cylinder in an ellipse.
Hint: you can do this by the same method used in Exercise 27 of Section 8.1.

3. Given two points F_1 and F_2 that are foci of an ellipse and a third point P on the ellipse, describe a geometric method (using straight edge and compass) for constructing the tangent line to the ellipse at P. *Hint:* think about the reflection property of ellipses.

4. Let C be a parabola with vertex V, and let P be any point on the parabola. Let R be the point where the tangent to the parabola at P intersects the axis of the parabola. (Thus the axis is the line RV.) Let Q be the point on RV such that PQ is perpendicular to RV. Show that V bisects the line segment RQ. How does this result suggest a geometric method for constructing a tangent to a parabola at a point on it, given the axis and vertex of the parabola?

5. A barrel has the shape of a solid of revolution obtained by rotating about its major axis the part of an ellipse lying between lines through its foci perpendicular to that axis. The barrel is 4 ft high and 2 ft in radius at its middle. What is its volume?

6. (a) Show that any straight line not passing through the origin can be written in polar form as

$$r = \frac{a}{\cos(\theta - \theta_0)},$$

where a and θ_0 are constants. What is the geometric significance of these constants?

(b) Let $r = g(\theta)$ be the polar equation of a straight line that does not pass through the origin. Show that

$$g^2 + 2(g')^2 - gg'' = 0.$$

(c) Let $r = f(\theta)$ be the polar equation of a curve, where f'' is continuous and $r \neq 0$ in some interval of values of θ. Let

$$F = f^2 + 2(f')^2 - ff''.$$

Show that the curve is turning toward the origin if $F > 0$ and away from the origin if $F < 0$. *Hint:* let $r = g(\theta)$ be the polar equation of a straight line tangent to the curve, and use part (b). How do f, f', and f'' relate to g, g', and g'' at the point of tangency?

7. **(Fast trip, but it might get hot)** If we assume that the density of the earth is uniform throughout, then it can be shown that the acceleration of gravity at a distance $r \le R$ from the centre of the earth is directed toward the centre of the earth and has magnitude $a(r) = rg/R$, where g is the usual acceleration of gravity at the surface ($g \approx 32$ ft/s^2), and R is the radius of the earth ($R \approx 3960$ mi). Suppose that a straight tunnel AB is drilled through the earth between any two points A and B on the surface, say Atlanta and Baghdad. (See Figure 8.55.) Suppose that a vehicle is constructed that can slide without friction or air resistance through this tunnel. Show that such a vehicle will, if released at one end of the tunnel, fall back and forth between A and B, executing simple harmonic motion with period $2\pi\sqrt{R/g}$. How many minutes will the round trip take? What is surprising here is that this period does not depend on where A and B are, or on the distance between them. *Hint:* Let the x-axis lie along the tunnel, with origin at the point closest to the centre of the earth. When the vehicle is at position with x-coordinate $x(t)$, its acceleration along the tunnel is the component of the gravitational acceleration along the tunnel, that is, $-a(r)\cos\theta$, where θ is the angle between the line of the tunnel and the line from the vehicle to the centre of the earth.

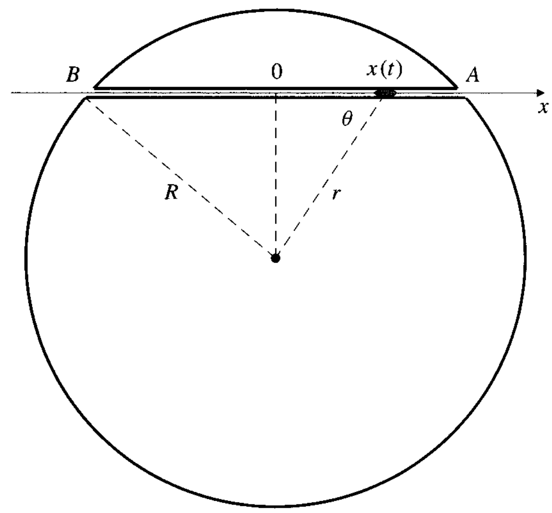
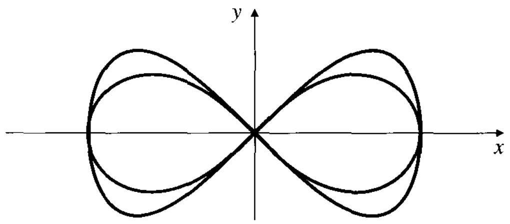


Figure 8.55

* 8. **(Search and Rescue)** Two coast guard stations pick up a distress signal from a ship and use radio direction finders to locate it. Station O observes that the distress signal is coming from the northeast (45° east of north), while station P, which is 100 miles north of station O, observes that the signal is coming from due east. Each station's direction finder is accurate to within $\pm 3°$.

(a) How large an area of the ocean must a rescue aircraft search to ensure that it finds the foundering ship?

(b) If the accuracy of the direction finders is within $\pm\varepsilon$, how sensitive is the search area to changes in ε when $\varepsilon = 3°$? (Express your answer in square miles per degree.)



9. Figure 8.56 shows the graphs of the parametric curve $x = \sin t$, $y = \frac{1}{2}\sin(2t)$, $0 \le t \le 2\pi$, and the polar curve $r^2 = \cos(2\theta)$. Each has the shape of an "∞." Which curve is which? Find the area inside the outer curve and outside the inner curve.